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SURFACES WITH GENERALIZED SECOND FUNDAMENTAL FORM IN L^2 ARE LIPSCHITZ MANIFOLDS

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Abstract

This paper focuses in the relationship between the class of surfaces with second fundamental form in L^2 and the class of Lipschitz surfaces (i.e., surfaces that are locally homeomorphic via a bilipschitz map to a flat disc). In particular we prove that graphs of $W^{2,2}(\mathbf{R}^2)$ functions are Lipschitz surfaces.

Introduction

For functions u, defined on a domain $\Omega \subset \mathbf{R}^2$, having locally square integrable partial derivatives up to order 2 (in the generalized sense), the Sobolev embedding theorems guarantee that u is locally Hölder continuous with any exponent $\alpha < 1$, and also that the gradient Du is locally in L^p for every $p < \infty$. There are, of course, examples illustrating that such u may not be locally Lipschitz—that is, Du need not be locally bounded in Ω. Since it gives some important insight into the nature of the singularities of general $W^{2,2}$ functions, we discuss a couple of particular examples in some detail.

Example 1. Let D be the disc of radius $\frac{1}{2}$ in $\mathbf{R}^2$, and let $u : D \to \mathbf{R}$ be defined by $u(x, y) = x \log|\log r|$, where $r = \sqrt{x^2+y^2}$. Direct computation shows that the Hessian D^2u is in $L^2(D)$; in fact $|D^2u| \leq Cr^{-1}|\log r|^{-1}$. On the other hand, Du is evidently unbounded on approach to the origin, in fact

$$u_x = \log|\log r| + O(|\log r|^{-1}) \quad \text{and} \quad u_y = O(|\log r|^{-1}) \quad \text{as } r \downarrow 0.$$

One can easily check that while Du has a singularity at 0, the unit normal $\nu = (1+|du|^2)^{-1/2}(-Du, 1)$ has limit $-e_1 = (-1, 0, 0)$ as $r \downarrow 0$ and the graph of u is a C^1 surface embedded in $\mathbf{R}^3$ with tangent plane normal

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to $-e_1$ at the origin. This example seems to indicate that while Du has singular behavior, nevertheless we might expect the graph of u to have some reasonable behavior. The second example, based on a modification of Example 1, shows that this fact is not obvious.

Example 2. Let D be the disc of radius $\frac{1}{2}$ in $\mathbf{R}^2$, and let $u : D \to \mathbf{R}$ be defined by $u(x, y) = x \log|\log r| \sin(\log|\log r|)$. In this case $|D^2 u| \leq C \log|\log r|(r|\log r|)^{-1}$, and therefore the Hessian is in $L^2(D)$. The gradient satisfies

$$u_x = \log|\log r| \sin(\log|\log r|) + O\left(\frac{\log|\log r|}{|\log r|}\right),$$

$$u_y = O\left(\frac{\log|\log r|}{|\log r|}\right) \quad \text{as } r \downarrow 0.$$

One can check that there is a sequence $R_k \uparrow \infty$ such that for each $\theta \in \mathbf{R}$ and each $\epsilon > 0$

$$\theta - \epsilon < u_x < \theta + \epsilon \quad \text{for } R_k^{-1} < r/r_k < R_k,$$

where $r_k \downarrow 0$ is the sequence of points such that $\log|\log r_k| \sin(\log|\log r_k|) = \theta$. We conclude that for every $K > 1$ there exists R_K such that for each $s \in (0, R_K)$ the graph $z = u(x, y)$ of u is close to the plane $x = \theta z$ in the annular region $K^{-1} < r/s < K$. Thus, roughly speaking, the graph is always close to a plane $L_s : z = \theta_s x$ in annular regions $K^{-1} < r/s < K$, but this plane changes slowly as s changes; furthermore the slope θ_s of the plane L_s oscillates (very slowly) between very large positive values and very large negative values.

Thus this graph fails to be C^1 (as it was for the example above). We can easily construct examples with much more singular behavior than the exhibited in Example 2. For instance, let $\widetilde{D}$ be the disc of radius $\frac{1}{4}$, (x_j, y_j) be a countable dense subset of $\widetilde{D}$ and for each $N \geq 1$ let

$$u_N = \sum_{j=1}^{N} 2^{-j}(x - x_j) \log|\log r_j| \sin(\log|\log r_j|),$$

where $r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$. Each u_N has singular behavior at each of the points $(x_j, y_j)_{j=1,\dots,N}$ like the singular behavior of Example 2. The sequence $\{u_N\}$ converges in the $W^{2,2}(\widetilde{D})$ norm to a function $u \in W^{2,2}(\widetilde{D})$ with a countable dense set of singular points.

Despite the pathologies presented in the examples above we show here that it is nevertheless true that the *graph*, $\mathcal{S} = \{(x, u(x)) : x \in \Omega \subset \mathbf{R}^2\}$, of a $W^{2,2}$ function u is a Lipschitz surface. Thus for each point

$X_0 = (x_0, u(x_0)) \in \mathscr{S}$, we can find $R > 0$, a domain $D \subset \mathbf{R}^2$, and a homeomorphism $\Phi : D \to S \cap B_R(X_0)$ such that

$$(*) \quad \begin{cases} |\Phi(x) - \Phi(y)| \le L|x - y| & \forall x, y \in D, \\ |\Phi^{-1}(X) - \Phi^{-1}(Y)| \le L|X - Y| & \forall X, Y \in \mathscr{S} \cap B_R(X_0), \end{cases}$$

where L is a positive constant.

Theorem. *Let $\Omega \subset \mathbf{R}^2$ be a strongly Lipschitz domain, $u \in W^{2,2}(\Omega, \mathbf{R})$ and $\mathscr{S} = \{(x, u(x)) : x \in \overline{\Omega}\}$. Then, there exist a domain $\Omega' \subset \mathbf{R}^2$ and a homeomorphism $\Phi : \Omega' \to \mathscr{S}$ so that $(*)$ holds with $L \le (1 + C\|u\|^2_{W^{2,2}(\Omega)})^{1/2}$. Further, the metric $g = (d\Phi)^* \circ d\Phi$ (i.e., the metric induced on Ω' by pulling back the metric of graph u induced by the Euclidean metric of $\mathbf{R}^n$) is comparable to the standard Euclidean metric of Ω' in the sense that*

$$\sup_{x \in \Omega'} |g_{ij}(x) - \delta_{ij}| \le C\|u\|^2_{W^{2,2}(\Omega)},$$

where g_{ij} are the components of g; thus $g_{ij}(x) = \langle \Phi_{|_x}(e_i), \Phi_{|_x}(e_j)\rangle$ for $i, j = 1, 2$. Here C is a constant that depends only on Ω.

Actually the main result in this direction is somewhat more general, beging applicable to a larger class of surfaces in $\mathbf{R}^n$. Given $\beta, \epsilon, \rho > 0$, let $\mathscr{T}_{\beta,\epsilon}(B_\rho(\zeta))$ denote the set of C^∞ embedded and connected surfaces $\mathscr{S}$ in $\mathbf{R}^n$, with $\partial\mathscr{S} \cap B_\rho(\zeta) = \varnothing$, and satisfying

$$\mathscr{H}^2(\mathscr{S} \cap B_\rho(\zeta)) \le \beta\rho^2 \quad \text{and} \quad \int_{\mathscr{S} \cap B_\rho(\zeta)} |A|^2 \, d\mathscr{H}^2 \le \epsilon^2.$$

Here A denotes the second fundamental form of $\mathscr{S}$, i.e., for $\zeta \in \mathscr{S}$, $A(\zeta)$ is the symmetric bilinear form on $T_\zeta\mathscr{S}$ with eigenvalues the principal curvatures of $\mathscr{S}$ at ζ. Let $\overline{\mathscr{T}_{\beta,\epsilon}(B_\rho(\zeta))}$ be the set of integer multiplicity varifolds $\underline{v}(\mathscr{S}, \theta)$ which in $B_\rho(\zeta)$ can be expressed as a measure theoretic limit of sequences $\{\mathscr{S}_k\}$, where $\mathscr{S}_k \in \mathscr{T}_{\beta,\epsilon}(B_\rho(\zeta))$. The main theorem in this setting is the following result:

Theorem. *For any $\beta > 0$, there exists $\epsilon_0 = \epsilon_0(\beta, n)$ so that if $\underline{v}(\mathscr{S}, \theta) \in \overline{\mathscr{T}_{\beta,\epsilon_0}(B_\rho(\zeta))}$ and $\zeta \in \mathscr{S}$, then*

$$\underline{v}(\mathscr{S} \llcorner B_{\rho/64}(\zeta)) = \sum_{i=1}^{N_\zeta} \underline{v}(\mathscr{D}_i \llcorner B_{\rho/64}(\zeta)),$$

where each $\mathscr{D}_i$ is the image of a disc in $\mathbf{R}^2$ via a bilipschitz map Φ_i, and where the decomposition is compatible with the multiplicity. Moreover for

$i = 1, \ldots, N_\zeta$,

$$\|(d\Psi_i)^* \circ (D\Psi_i) - \imath\|_{L_\infty} \leq C\epsilon_0^2, \quad \textit{and} \quad \operatorname{Lip}\Psi_i, \operatorname{Lip}\Psi_i^{-1} \leq 1 + C\epsilon_0^2.$$

Here $(\ )^*$ *denotes the adjoint, and* $\imath$ *denotes the identity transformation on* $\mathbf{R}^2$.

Now, we would like to indicate how to prove the first theorem as a corollary of this last theorem. Since Ω is a strongly Lipschitz domain, Calderon's extension theorem asserts that there exists a function $v \in W^{2,2}(\mathbf{R}^2, \mathbf{R})$ so that

$$v_{|\Omega} = u, \quad v = 0 \text{ on } \mathbf{R}^2 \backslash B_R(0), \quad \text{and} \quad \|v\|_{W^{2,2}(\mathbf{R}^2)} \leq C\|u\|_{W^{2,2}(\Omega)},$$

for some R large enough, and where C only depends on Ω. For $\lambda \in (0, 1]$, let $\mathcal{S}_\lambda = \{(x, v_\lambda(x)) : x \in \mathbf{R}^2\}$, where $v_\lambda(x) = \lambda v(x)$. Since $v_\lambda \in W_0^{2,2}(B_{2R}(0) \cap \mathbf{R}^2)$, then $v_\lambda \in C^0$ (see [2]) and $\mathcal{S}_\lambda = \operatorname{graph} v_\lambda$ is C^0 embedded in $\mathbf{R}^3$. There exists a sequence of functions $v_j \in W^{2,2}(\mathbf{R}^2) \cap C_C^\infty(B_{2R}(0) \cap \mathbf{R}^2)$ that approximate v_λ in the $W^{2,2}$ norm. In particular for j large enough

$$\|v_j\|_{W^{2,2}(\mathbf{R}^2)} \leq 2\|v_\lambda\|_{W^{2,2}(\mathbf{R}^2)} \leq 2\lambda\|v\|_{W^{2,2}(\mathbf{R}^2)},$$

and if A_j denotes the second fundamental form of $\mathcal{S}_j = \operatorname{graph} v_j$, then we have

$$\int_{\mathcal{S}_j} |A_j|^2 \, d\mathcal{H}^2 \leq C \int_{\mathbf{R}^2} |D^2 v_j|^2 \leq C_0 \lambda^2 \|u\|^2_{W^{2,2}(\Omega)}.$$

Moreover there exists $K > 0$ so that for all $j \geq 1$, $\sup_{\mathbf{R}^2} |v_j| \leq K$. Choosing $\rho > 0$ large enough so that $\operatorname{graph} v_{j|_{B_{2\mathbf{R}}(0) \cap \mathbf{R}^2}} \subset B_\rho(0)$, the monotonicity formula (see §4) guarantees that for $r \geq \rho$

$$\rho^{-2}\mathcal{H}^2(\mathcal{S}_j \cap B_\rho(0)) \leq C\left(r^{-2}\mathcal{H}^2(\mathcal{S}_j \cap B_r(0)) + \int_{\mathcal{S}_j \cap B_r(0)} |A_j|^2 \, d\mathcal{H}^2\right) \leq \beta.$$

Choosing λ small enough so that $C_0\lambda^2\|u\|^2_{W^{2,2}(\Omega)} \leq \varepsilon_0^2$, where $\varepsilon_0 = \varepsilon_0(\beta)$ is as in the theorem above we conclude that for j large enough $\mathcal{S}_j \cap B_\rho(0) \in \mathcal{T}_{\beta,\varepsilon}(B_\rho(0))$ and therefore $\mathcal{S}_\lambda \in \overline{\mathcal{T}_{\beta,\varepsilon}(B_\rho(0))}$. Thus applying the previous result combined with the fact that $\mathcal{S}_\lambda$ is C^0 embedded in $\mathbf{R}^3$ we are led to that there exists a bilipschitz homeomorphism $\Phi_\lambda : \Omega' \to \mathcal{S}_\lambda$ so that

$$\operatorname{Lip}\Phi_\lambda, \operatorname{Lip}\Phi_\lambda^{-1} \leq 1 + C\|v_\lambda\|^2_{W^{2,2}(\mathbf{R}^2)},$$

and

$$\|(d\Phi_\lambda)^* \circ (d\Phi_\lambda) - \imath\|_{L^\infty(\Omega')} \leq C\|v_\lambda\|^2_{W^{2,2}(\mathbf{R}^2)}.$$

Let $R_\lambda(x^1, x^2, x^3) = (x^1, x^2, \lambda^{-1}x^3)$ and define $\Phi = R_\lambda \circ \Phi_\lambda$. One easily checks that Φ is a suitable bilipschitz homeomorphism onto graph u.

The second theorem gives some insight on the structure of integer multiplicity 2-dimensional varifolds with generalized second fundamental form in L^2. Specifically, for an open domain $U \subset \mathbf{R}^n$ with $0 \in U$, let $\mathcal{T}(U)$ denote the set of multiplicity one 2-dimensional varifolds without boundary, $\underline{v}(\mathcal{S})$, with C^∞ connected support in U, containing 0 and which have uniform local bounds in U on their areas and on the L^2 norms of their second fundamental form. Let $\overline{\mathcal{T}(U)}$ be the set of $\underline{v}(\mathcal{S}, \theta)$ which in U, can be expressed as the measure theoretic limit of sequences $\{\underline{v}(\mathcal{S}_k)\}$, where $\underline{v}(\mathcal{S}_k) \in \mathcal{T}(U)$. That is, we assume, that for each compact $K \subset U$ there is a constant C_K such that $\mathcal{H}^2(\mathcal{S}_k \cap K) \le C_K$, $\int_{\mathcal{S}_k \cap K} |A_k|^2 \, d\mathcal{H}^2 \le C_K$ and $\int_{\mathcal{S}_k} f \, d\mathcal{H}^2 \to \int_{\mathcal{S}} f \, d\mu$ for each fixed continuous $f \colon U \to \mathbf{R}$ with compact support in U. Under these conditions $\mu = \mathcal{H}^2 \llcorner \theta$, where θ is a positive integer-valued function; $\{\mathcal{S}_k\}$ converges to $\mathcal{S}$ in the Hausdorff distance sense and the generalized second fundamental form A of $\mathcal{S}$ (see [3]) is well defined and in L^2 with respect to the 2-dimensional Hausdorff measure on $\mathcal{S}$. Then we have

Corollary. *For $\underline{v}(\mathcal{S}, \theta) \in \overline{\mathcal{T}(U)}$, there are finitely many points $\zeta_1, \cdots, \zeta_p \in \mathcal{S}$ so that for all $\zeta \in \mathcal{S} \setminus \{\zeta_1, \cdots, \zeta_p\}$ there exists $r(\zeta) > 0$ such that if $0 < r \le r(\zeta)$, then*

$$\underline{v}(\mathcal{S} \llcorner B_r(\zeta)) = \sum_i^{N_\zeta} \underline{v}(\mathcal{D}_i \llcorner B_r(\zeta)),$$

where each $\mathcal{D}_i$ is a bilipschitz image of a disc in $\mathbf{R}^2$, and where the decomposition is compatible with the multiplicity.

In order to prove the main theorem we initially focus our attention on a special type of neighborhoods, the *quasirectangles* which behave very much like rectangles in $\mathbf{R}^2$, in the appropriate sense. In particular they admit parameterizations that are bilipschitz with respect to their intrinsic distance. Then using the Approximate Graphical Decomposition Lemma [11], [12], we prove that if $\zeta \in \mathcal{S}$ and $\int_{\mathcal{S} \cap B_\rho(\zeta)} |A|^2 \, d\mathcal{H}^2$ is small enough, there exists a *quasirectangle* in $\mathcal{S} \cap B_\rho(\zeta)$ containing ζ, where the euclidean distance and the intrinsic distance are equivalent. The result about the equivalence of the Euclidean and the intrinsic distances also follows from work of G. David and S. Semmes concerning surfaces with unit normal having small BMO norm; see [7], [8], [9].

We would like to emphasize that all the constants C which appear in this paper only depend on n, the dimension of the ambient space, and in particular do not depend on the surface. We always assume $C \geq 1$.

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2. Quasirectangles

Most of the technical content of the theorem lies in the proof of the fact that, intrinsically, *quasirectangles* are bilipschitz surfaces. This section is devoted to the study of this type of neighborhoods.

Definition 2.1. Let $\alpha \in (0, \frac{1}{8})$ and let $\Sigma \subset \mathbf{R}^n$ be diffeomorphic, via a C^∞ diffeomorphism, to the unit square $[0, 1] \times [0, 1]$. We say that Σ is an *α-quasirectangle* if the following conditions hold:

(i) $\int_\Sigma |A|^2 \, d\mathscr{H}^2 \leq \alpha^2$.

(ii) There exists a 2-dimensional subspace $L \subset \mathbf{R}^n$ (spanned by τ_1, τ_2 with $|\tau_1| = |\tau_2| = 1$, $\langle \tau_1, \tau_2 \rangle = 0$) so that $\partial\Sigma$ projects simply onto L and

$$\sup_{\zeta \in \partial\Sigma} |\tau_1(\zeta) \wedge \tau_2(\zeta) - \eta| \leq \alpha,$$

$$\eta = \tau_1 \wedge \tau_2 = \tau_1(\zeta_0) \wedge \tau_2(\zeta_0) \text{ for some } \zeta_0 \in \partial\Sigma,$$

where $\tau_1(\zeta), \tau_2(\zeta)$ form an orthonormal basis for $T_\zeta\Sigma$.

(iii) There exist a rectangle $Q \subset \mathbf{R}^2$ with

$$1 \leq \frac{\text{length of the longer side of } Q}{\text{length of the shorter side of } Q} \leq 2,$$

and a smooth map $f: \mathbf{R}^2 \to L$ with $f(Q) = R$, where R is the compact region of L bounded by the orthogonal projection Λ of $\partial\Sigma$ onto L, so that

$$\sup_{x \in \mathbf{R}^2} \|(df^* \circ df)_x - \iota_x\| \leq \alpha^2,$$

where $(\ )^*$ denotes the adjoint, and ι is the identity transformation of $\mathbf{R}^2$. Σ will be referred to simply as a quasirectangle if it is an α-quasirectangle with $\alpha \in (0, \frac{1}{8})$. We denote by $\nu(\zeta)$ the 2-vector $\tau_1(\zeta) \wedge \tau_2(\zeta)$ orthogonal to $T_\zeta\Sigma$.

Remarks.

2.1. Notice that, since $|df_x(\tau)| \leq \sqrt{1 + \alpha^2} < 1 + \alpha^2$ for each unit vector τ, by integration along straight line segments we see that f satisfies the

Lipschitz condition

$$|f(x) - f(y)| \le (1+\alpha^2)|x-y| \le \tfrac{9}{8}|x-y|, \quad \forall x, y \in \mathbf{R}^2.$$

Assume that $L = \mathbf{R}^2$. Then $\det df \ne 0$, and hence f would be a covering map of $\mathbf{R}^2$. This implies, by a monodromy argument that f is $1:1$. Thus (even in the case where L is arbitrary), we have that f is a diffeomorphism. Also, condition (iii) above implies that the inverse h of f satisfies $\sup_{x\in L} \|(dh^* \circ dh)_x - \iota_x\| \le 2\alpha^2$, and hence that

$$|h(x) - h(y)| \le (1+2\alpha^2)|x-y| \le \tfrac{9}{8}|x-y|, \quad \forall x, y \in L.$$

2.2. Since $\partial\Sigma$ projects simply onto L, the curve Λ of the above definition is the diffeomorphic image of the boundary of the unit square. Since $\operatorname{osc}_{\partial\Sigma} \nu \le 2\sup_{\partial\Sigma} |\nu - \eta| \le 2\alpha$, there exist a neighborhood W of Λ in L and a function $w \in C^\infty(L, L^\perp)$ so that $V = \operatorname{graph} w_{|W\cap R}$ is a boundary neighborhood of Σ and $\partial\Sigma = \operatorname{graph} w_{|\partial R}$. Here for $A \subset L$, $\operatorname{graph} w_{|A} = \{x + w(x) : x \in A\}$. For $\zeta \in V \subset \Sigma$ there exists $x \in W$ so that $\zeta = x + w(x)$, and the 2-vector normal to Σ at ζ can be expressed as

$$\nu(\zeta) = \frac{\tau_1 \wedge \tau_2 + D_{\tau_1} w \wedge \tau_2 + \tau_1 \wedge D_{\tau_2} w + D_{\tau_1} w \wedge D_{\tau_2} w}{(1 + |Dw|^2 + |D_{\tau_1} w \wedge D_{\tau_2} w|^2)^{1/2}}.$$

The fact that $\eta = \nu(\zeta_0)$ for some $\zeta_0 \in \partial\Sigma$, guarantees that we can find such a w satisfying

$$\sup_L |Dw| \le \operatorname*{osc}_L Dw \le 4\operatorname{osc}_{\partial\Sigma} \nu < \tfrac{1}{2}.$$

Lemma 2.1. *If Σ is a quasirectnagle with corresponding rectangle $Q \subset \mathbf{R}^2$ as in Definition* 2.1, *then* $\operatorname{diam}\Sigma \le C \operatorname{diam} Q$ *and* $\mathscr{H}^2(\Sigma) \le C(\operatorname{diam} Q)^2$, *where* $\operatorname{diam} Q$ *denotes the diameter of* Q, *and* $\mathscr{H}^2$ *denotes the* 2*-dimensional Hausdorff measure.*

Proof. By the first variation formula

$$\int_\Sigma \operatorname{div}_\Sigma \Phi = \int_\Sigma \langle \underline{H}, \Phi\rangle + \int_{\partial\Sigma} \langle v, \Phi\rangle,$$

where the notation is as follows: $\underline{H}$ is the mean curvature vector of Σ, $|\underline{H}| = |\operatorname{trace} A|$, Φ is any Lipschitz vector field defined in a neighborhood of Σ, v is the outward unit conormal vector of $\partial\Sigma$, and $\operatorname{div}_\Sigma \Phi$ is the tangential divergence of Φ. Setting $\Phi(\zeta) = \zeta - \zeta_0$, where ζ_0 is a point of $\partial\Sigma$ we deduce that

$$2\mathscr{H}^2(\Sigma) = \int_\Sigma \langle \underline{H}, \zeta - \zeta_0\rangle + \int_{\partial\Sigma} \langle v, \zeta - \zeta_0\rangle.$$

By the definition of quasirectangle we have that

$$\begin{aligned}\mathscr{H}(\partial\Sigma) &\le \left(1+\sup_L |Dw|^2\right)\mathscr{H}(\Lambda)\\ &\le \tfrac{5}{4}\operatorname{Lip} f|\partial Q| \le \tfrac{5}{4}\cdot\tfrac{9}{8}\cdot 4\cdot \operatorname{diam} Q \le 6\operatorname{diam} Q,\end{aligned}$$

where $|\partial Q|$ denotes the length of ∂Q, and $\mathscr{H}$ denotes the 1-dimensional Hausdorff measure. Moreover since $\operatorname{diam}\partial\Sigma \le \frac{1}{2}\mathscr{H}(\partial\Sigma)$, we have

$$2\mathscr{H}^2(\Sigma) \le \operatorname{diam}\Sigma\int_\Sigma |\underline{H}| + \mathscr{H}(\partial\Sigma)\operatorname{diam}\partial\Sigma \le \operatorname{diam}\Sigma\int_\Sigma |\underline{H}| + 18(\operatorname{diam} Q)^2.$$

Applying the Cauchy-Schwarz inequality on the right yields

$$\begin{aligned}2\mathscr{H}^2(\Sigma) &\le \operatorname{diam}\Sigma(\mathscr{H}^2(\Sigma))^{1/2}\left(\int_\Sigma |\underline{H}|^2\right)^{1/2} + 18(\operatorname{diam} Q)^2\\ &\le 2\operatorname{diam}\Sigma(\mathscr{H}^2(\Sigma))^{1/2}\left(\int_\Sigma |A|^2\right)^{1/2} + 18(\operatorname{diam} Q)^2.\end{aligned}$$

We need to estimate $\operatorname{diam}\Sigma$ in terms of $\operatorname{diam} Q$ so: either $\operatorname{diam}\Sigma \le 4\mathscr{H}(\partial\Sigma) \le 24\operatorname{diam} Q$ or $\operatorname{diam}\Sigma > 4\mathscr{H}(\partial\Sigma)$. In the first case we conclude that

$$\begin{aligned}2\mathscr{H}^2(\Sigma) &\le 2\cdot 24\cdot\tfrac{1}{8}\operatorname{diam} Q(\mathscr{H}^2(\Sigma))^{1/2} + 18(\operatorname{diam} Q)^2\\ &\le 6\operatorname{diam} Q(\mathscr{H}^2(\Sigma))^{1/2} + 18(\operatorname{diam} Q)^2\\ &\le \mathscr{H}^2(\Sigma) + 27(\operatorname{diam} Q)^2.\end{aligned}$$

In the second case there exists $\zeta_1\in\Sigma$ such that $\operatorname{dist}(\zeta_1, \partial\Sigma) \ge \frac{1}{16}\operatorname{diam}\Sigma$. We apply the first variation formula to $\Phi(\zeta) = |X|_\sigma^{-2}X$ where $X = \zeta - \zeta_1$, $0<\sigma<\frac{1}{16}\operatorname{diam}\Sigma < \rho$, $|X|_\sigma = \max\{|\zeta-\zeta_1|, \sigma\}$, and $B_\rho(\zeta_1)\cap\partial\Sigma = \emptyset$. Letting $\sigma\downarrow 0$ we have the identity

$$\pi + \int_\Sigma\left(\frac{1}{4}\underline{H} + \frac{(X)^\perp}{|X|^2}\right)^2 = \frac{1}{2}\int_{\partial\Sigma}\left\langle v, \frac{X}{|X|^2}\right\rangle + \frac{1}{16}\int_\Sigma |\underline{H}|^2,$$

where $(\ )^\perp$ denotes the projection onto the normal space to Σ. Hence

$$\begin{gathered}\pi \le \frac{8}{\operatorname{diam}\Sigma}\mathscr{H}(\partial\Sigma) + \frac{1}{8}\int_\Sigma |A|^2\, d\mathscr{H}^2,\\ 3\operatorname{diam}\Sigma \le 8\mathscr{H}(\partial\Sigma) + \frac{1}{64}\operatorname{diam}\Sigma.\end{gathered}$$

The above inequality implies $\operatorname{diam}\Sigma \le 4\mathscr{H}(\partial\Sigma)$, which contradicts our original assumption. Therefore we always have $\operatorname{diam}\Sigma \le 4\mathscr{H}(\partial\Sigma) \le 24\operatorname{diam} Q$ which implies $\mathscr{H}^2(\Sigma) \le C(\operatorname{diam} Q)^2$.

The following lemma is the key technical ingredient of the proof of the main theorem. It shows that a quasirectangle with nice boundary, in a suitable sense, is the image of its associated rectangle by means of a map that is bilipschitz with respect to the intrinsic distance.

Main Lemma. *There is a fixed constant* $\varepsilon_0 > 0$ *such that if* $\varepsilon \le \varepsilon_0$, *if* $\Sigma^{(0)}$ *is an* ε-*quasirectangle* (*satisfying conditions* (i), (ii), (iii) *above with* $\Sigma^{(0)}, Q^{(0)}, L^{(0)}, \eta^{(0)}, f^{(0)}, \varepsilon$ *in place of* $\Sigma, Q, L, \eta, f, \alpha$ *respectively*) *and if* $\mathscr{H}(\partial\Sigma^{(0)}) \int_{\partial\Sigma^{(0)}} |A|^2 \le \varepsilon^2$, *then there exists a map* Φ *of* $Q^{(0)}$ *onto* $\Sigma^{(0)}$ *such that*

$$(1 + C\varepsilon^2)^{-1}|x - y| \le d(\Phi(x), \Phi(y)) \le (1 + C\varepsilon^2)|x - y| \quad \forall x, y \in Q^{(0)},$$

and

$$\|(d\Phi)^* \circ (d\Phi) - \imath\|_{L^\infty(Q^{(0)})} \le C\varepsilon^2,$$

where $d(\cdot, \cdot)$ *denotes the intrinsic distance measured in* $\Sigma^{(0)}$.

Remark 2.3. Note that the additional hypothesis $\mathscr{H}(\partial\Sigma^{(0)}) \int_{\partial\Sigma^{(0)}} |A|^2 \le \varepsilon^2$ guarantees the part of the definition of ε-quasirectangle which requires $\operatorname{osc}_{\partial\Sigma^{(0)}} \nu \le \varepsilon$.

Proof. The main ideas of the proof are: give a procedure for subdividing $Q^{(0)}$ into six subrectangles $Q_1^{(1)}, \cdots, Q_6^{(1)}$ and $\Sigma^{(0)}$ into six quasirectangles $\Sigma_1^{(1)}, \cdots, \Sigma_6^{(1)}$ which correspond to $Q_1^{(1)}, \cdots, Q_6^{(1)}$ respectively as in Definition 2.1; show that this construction can be iterated.

First we note that, by Remark 2.2, there is a smooth function $w^{(0)}: L^{(0)} \to (L^{(0)})^\perp$ and a neighborhood $W^{(0)}$ of $\Lambda^{(0)} = f^{(0)}(\partial Q^{(0)})$ so that graph $w^{(0)}_{|W^{(0)} \cap R^{(0)}} = V^{(0)}$ for some boundary neighborhood $V^{(0)}$ of $\Sigma^{(0)}$, $\partial\Sigma^{(0)} = \operatorname{graph} w^{(0)}_{|\partial R^{(0)}}$, and

$$\sup_{L^{(0)}} |Dw^{(0)}| \le \operatorname*{osc}_{L^{(0)}} Dw^{(0)} \le 4 \operatorname*{osc}_{\partial\Sigma^{(0)}} \nu \le 4\varepsilon.$$

Let $s_1(Q^{(0)}), s_3(Q^{(0)})$ be the edges of $Q^{(0)}$ parallel to the x-axis, labelled so that the y-coordinate of $s_1(Q^{(0)})$ is less than the y-coordinate of $s_3(Q^{(0)})$, and let $s_2(Q^{(0)}), s_4(Q^{(0)})$ be the edges of $Q^{(0)}$ parallel to the y-axis, labelled so that the x-coordinate of $s_2(Q^{(0)})$ is less than the x-coordinate of $s_4(Q^{(0)})$. Now we describe the subdivision of $Q^{(0)}$. Without loss of generality we assume that $|s_2(Q^{(0)})| \le |s_1(Q^{(0)})|$ where $|s_j(Q^{(0)})|$ means length of the edge $s_j(Q^{(0)})$. Let x_1, x_2 be the x-coordinates of the points $\frac{1}{3}, \frac{2}{3}$ of the way along the edge $s_1(Q^{(0)})$, and y_1 the y-coordinate of the midpoint of the edge $s_2(Q^{(0)})$. We slice $Q^{(0)}$ in lines

$\{x = \lambda_1\}$, $\{x = \lambda_2\}$, and $\{y = \lambda_3\}$, where λ_j is to be chosen in the interval $I_j = (x_j - \frac{1}{30}|s_1(Q^{(0)})|, x_j - \frac{1}{30}|s_1(Q^{(0)})|)$ for $j = 1, 2$, and λ_3 is to be chosen in the interval $I_3 = (y_1 - \frac{1}{30}|s_2(Q^{(0)})|, y_1 - \frac{1}{30}|s_2(Q^{(0)})|)$.

We shall make the actual choices of $\lambda_1, \lambda_2, \lambda_3$ shortly, but for the moment we observe that the subrectangles $Q_1^{(1)}, \cdots, Q_6^{(1)}$ obtained by so slicing $Q^{(0)}$ satisfy

$$\begin{aligned} &\tfrac{4}{15}|s_1(Q^{(0)})| \le |s_1(Q_k^{(1)})| \le \tfrac{6}{15}|s_1(Q^{(0)})|, \\ &\tfrac{7}{15}|s_2(Q^{(0)})| \le |s_2(Q_k^{(1)})| \le \tfrac{8}{15}|s_1(Q^{(0)})|, \end{aligned} \tag{1}$$

for $k = 1, \cdots, 6$. Hence for $k = 1, \cdots, 6$

$$\operatorname{diam}(Q_k^{(1)}) \le \tfrac{8}{15} \operatorname{diam} Q^{(0)},$$

and

$$\frac{1}{2} \le \frac{1}{2}\frac{|s_1(Q^{(0)})|}{|s_2(Q^{(0)})|} \le \frac{|s_1(Q_k^{(1)})}{|s_2(Q_k^{(1)})|} \le \frac{6}{7}\frac{|s_1(Q^{(0)})|}{|s_2(Q^{(0)})|} \le \frac{12}{7} \le 2.$$

It was precisely in order to arrange this property that we chose to subdivide $Q^{(0)}$ into six pieces, rather than into four.

Let $h^{(0)} = (f^{(0)})^{-1} : L^{(0)} \to \mathbf{R}^2$ and note that by Remark 2.1 we have

$$|h^{(0)}(x) - h^{(0)}(y)| \le (1 + 2\varepsilon^2)|x - y| \quad \text{for all } x, y \in R^{(0)},$$

and $h^{(0)}$ is C^∞ because $df^{(0)}$ is nonsingular at each point. Let $p^{(0)}$ be the orthogonal projection onto $L^{(0)}$, and consider the slices

$$\begin{aligned} \Gamma_{i,\lambda} &= \{\zeta \in \Sigma^{(0)} : \langle e_1, h^{(0)}(p^{(0)}\zeta)\rangle = \lambda\}, \qquad \lambda \in I_i,\ i = 1, 2, \\ \Gamma_{3,\lambda} &= \{\zeta \in \Sigma^{(0)} : \langle e_2, h^{(0)}(p^{(0)}\zeta)\rangle = \lambda\}, \qquad \lambda \in I_3, \end{aligned}$$

where e_1, e_2 are the canonical basis for $\mathbf{R}^2$. Since $h^{(0)} \circ p^{(0)}$ is smooth, Sard's theorem guarantees that $\Gamma_{i,\lambda}$ is a finite union of smooth Jordan arcs and closed Jordan curves for almost all λ, with the Jordan arcs having endpoints which project under $p^{(0)}$ to $\Lambda^{(0)} = \partial R^{(0)}$.

We have established the notation needed in the proof, which at this point can be divided in four distinct parts;

Part 1. We shall prove that it is possible to select $\lambda_i \in I_i$ for $i = 1, 2, 3$ such that each Γ_{i,λ_i} is a union of smooth Jordan curves and arcs, one of which is a Jordan arc γ_i with endpoints in $\partial\Sigma^{(0)}$ and

$$\operatorname*{osc}_{\gamma_i} \nu \le C\left(\int_{\Sigma^{(0)}} |A|^2\right)^{1/2} \le C\varepsilon.$$

Part 2. Assuming Part 1 we see that $\gamma_1, \gamma_2, \gamma_3$ divide $\Sigma^{(0)}$ into six pieces $\Sigma_1^{(1)}, \cdots, \Sigma_6^{(1)}$ each of them diffeomorphic to the unit square $[0, 1] \times [0, 1]$ and where the labelling is such that $Q_k^{(1)}$ corresponds to $\Sigma_k^{(1)}$ in the natural way. We then choose ε small enough so that if $\Sigma^{(0)}$ is an ε-quasirectangle we can guarantee that each one of the resulting $\Sigma_k^{(1)}$ is a $\varepsilon_k^{(1)}$-quasirectangle with $\varepsilon_k^{(1)} < \frac{1}{8}$. In particular, for each $k = 1, \cdots, 6$ we need to exhibit a plane $L_k^{(1)}$ and construct a function $f_k^{(1)}: \mathbf{R}^2 \to L_k^{(1)}$ satisfying conditions (ii) and (iii) from Definition 2.1.

Part 3. We shall prove that it is possible to choose ε, as in Part 2, so that the construction described in Parts 1 and 2 can be iterated arbitrarily many times. This fact is a consequence of the properties required from the curves $\gamma_1, \gamma_2, \gamma_3$ described in Part 1.

Part 4. From Part 3 we conclude that $\Sigma^{(0)}$ can be partitioned into arbitrarily small quasirectangles. The construction of the map $\Phi: Q^{(0)} \to \Sigma^{(0)}$ becomes then straightforward.

Part 1. Let $g_1, g_2, g_3: \Sigma^{(0)} \to \mathbf{R}$ be defined by $g_i(\zeta) = \langle e_1, h^{(0)} \circ p^{(0)}(\zeta)\rangle$ for $i = 1, 2$ and $g_3(\zeta) = \langle e_2, h^{(0)} \circ p^{(0)}(\zeta)\rangle$. By the co-area formula we have

$$\begin{aligned}\int_{I_i} \mathscr{H}(\Gamma_{i,\lambda})\, d\lambda &= \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_i\}} |\nabla^{\Sigma^{(0)}} g_i|\, d\mathscr{H}^2 \\ &\le \operatorname{Lip} h^{(0)} \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_i\}} d\mathscr{H}^2 \\ &\le \tfrac{9}{8}\mathscr{H}^2(\Sigma^{(0)}) \le C(\operatorname{diam} Q^{(0)})^2,\end{aligned}$$

by virtue of Lemma 2.1. For each $i = 1, 2, 3$ there exists a set $I_i' \subset I_i$ so that $|I_i'| \ge \frac{15}{16}|I_i|$ and $\forall \lambda \in I_i'$, $\mathscr{H}(\Gamma_{i,\lambda}) \le C \operatorname{diam} Q^{(0)}$. Let I_i^* be a subinterval of I_i of length $\frac{1}{4}|I_i|$ satisfying,

$$\int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_i^*\}} |A|^2 \le 2 \inf_{\widetilde{I}_i} \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in \widetilde{I}_i\}} |A|^2$$

over all subintervals $\widetilde{I}_i$ of I_i with length $\frac{1}{4}|I_i|$. By the co-area formula

$$\begin{aligned}\int_{I_i^*}\int_{\Gamma_{i,\lambda}} |A|^2\, d\mathscr{H}\, d\lambda &= \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_i^*\}} |\nabla^{\Sigma^{(0)}} g_i||A|^2 \\ &\le C \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_i^*\}} |A|^2,\end{aligned}$$

hence there exists a set $(I_i^*)' \subset I_i^*$ so that $|(I_i^*)'| \geq \frac{1}{2}|I_i^*|$ and $\forall\lambda \in (I_i^*)'$

$$\int_{\Gamma_{i,\lambda}} |A|^2 \leq \frac{C}{\operatorname{diam} Q^{(0)}} \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_i^*\}} |A|^2$$
$$\leq \frac{C}{\operatorname{diam} Q^{(0)}} \inf_{\widetilde{I}_i} \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in\widetilde{I}_i\}} |A|^2 .$$

Since $|I_i'| \geq \frac{15}{16}|I_i|$ and $|(I_i^*)'| \geq \frac{1}{2}|I_i^*| = \frac{1}{8}|I_i|$, we conclude that there exists a set $J_i \subset I_i$ so that $|J_i| \geq \frac{1}{16}|I_i|$ and $\forall\lambda \in J_i$

$$\mathscr{H}(\Gamma_{i,\lambda}) \int_{\Gamma_{i,\lambda}} |A|^2 \leq C \inf_{\widetilde{I}_i} \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in\widetilde{I}_i\}} |A|^2 .$$

In particular for $\lambda \in J_i$

$$\begin{aligned} &\mathscr{H}(\Gamma_{i,\lambda}) \int_{\Gamma_{i,\lambda}} |A|^2 \\ &\leq C \min\left\{ \int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_{i,1}\}} |A|^2,\ \int_{\{\zeta\in\sigma^{(0)}\,:\, g_i(\zeta)\in I_{i,2}\}} |A|^2 \right\}, \end{aligned} \tag{2}$$

where $I_{i,1}$, $I_{i,2}$ are the subintervals of I_i of length $\frac{1}{4}|I_i|$ which lie at opposite extremes of I_i. Since g_i and $\Sigma^{(0)}$ are smooth, Sard's theorem ensures that we can select all of these λ to be such that $\gamma_{i,\lambda}$ is a finite union of smooth closed Jordan curves and smooth Jordan arcs with endpoints in $\partial\Sigma^{(0)}$. Actually since for $i = 1, 2$ and $\lambda \in J_i$, $f^{(0)}(\{(\lambda, y) : y \in \mathbf{R}\})$ are smooth curves, each of them passing through $\partial R^{(0)}$ in exactly two points, and since $p^{(0)}$ projects $\partial\Sigma^{(0)}$ simply onto $L^{(0)}$, for each such λ, there is exactly one Jordan arc in the above union, which we call $\Gamma^*_{i,\lambda}$ with endpoints on $\partial\Sigma^{(0)}$. A similar result holds for $\lambda \in J_3$, $f^{(0)}(\{(x, \lambda) : x \in \mathbf{R}\})$ and $\Gamma^*_{3,\lambda}$. Notice that, since $|A| = |\nabla^{\Sigma^{(0)}}\nu|$, by integration along the Jordan arc $\Gamma^*_{i,\lambda}$ we have

$$\operatorname*{osc}_{\Gamma^*_{i,\lambda}} \nu \leq \int_{\Gamma^*_{i,\lambda}} |A| \leq \left(\mathscr{H}(\Gamma_{i,\lambda}) \int_{\Gamma_{i,\lambda}} |A|^2 \right)^{1/2},$$

hence by (2)

$$
\begin{aligned}
\operatorname*{osc}_{\Gamma^*_{i,\lambda}} \nu &\le C \min\left\{ \left(\int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_{i,1}\}} |A|^2 \right)^{\frac{1}{2}}, \right. \\
&\qquad \left. \left(\int_{\{\zeta\in\Sigma^{(0)}\,:\, g_i(\zeta)\in I_{i,2}\}} |A|^2 \right)^{\frac{1}{2}} \right\} \qquad (3)\\
&\le C\varepsilon .
\end{aligned}
$$

For $\varepsilon > 0$ small $\Gamma^*_{i,\lambda}$ projects simply into $R^{(0)}$. Furthermore since $\sup_{x\in Q^{(0)}} \|(df_o^* \circ df_o)_x - \iota_x\| \le \varepsilon^2$, the intersection with $\partial R^{(0)}$ at the endpoints is transversal and so are the intersections of $\Gamma^*_{1,\lambda}$ and $\Gamma^*_{2,\lambda}$ with $\Gamma^*_{3,\lambda}$ (in fact almost orthogonal for ε small enough). Therefore we can select $\Gamma^*_{i,\lambda}$ to be the required curves γ_i, $i = 1, 2, 3$, so that in particular $\operatorname{osc}_{\gamma_i} \nu \le C\varepsilon$, and the γ_i's can be used to subdivide $\Sigma^{(0)}$ into six pieces $\Sigma_1^{(1)}, \cdots, \Sigma_6^{(1)}$ diffeomorphic to the unit square $[0, 1]\times[0, 1]$, where the labelling is so that $Q_k^{(1)}$ corresponds to $\Sigma_k^{(1)}$ in the natural way.

Part 2. In Part 1 we proved that $\partial\Sigma_k^{(1)}$ projects simply onto $L^{(0)}$, that $f^{(0)}(Q_k^{(1)})$ is the compact region of $L^{(0)}$ bounded by $p^{(0)}(\partial\Sigma_k^{(1)})$, that

$$
\operatorname*{osc}_{\partial\Sigma_k^{(1)}} \nu \le C\left(\int_{\Sigma^{(0)}} |A|^2 \right)^{1/2} + \operatorname*{osc}_{\partial\Sigma^{(0)}} \nu \le C\varepsilon, \qquad (4)
$$

and that

$$
\operatorname*{osc}_{\partial\Sigma_k^{(1)}} |\nu - \eta^{(0)}| \le \operatorname*{osc}_{\partial\Sigma_k^{(1)}} \nu + \operatorname*{osc}_{\partial\Sigma^{(0)}} \nu \le 2C\varepsilon,
$$

for $k = 1, \cdots, 6$. By the same argument used in Remark 2.2, to prove the existence of $w\colon L \to L^\perp$, we establish that for each k there is a smooth function $w_k^{(0)}\colon L^{(0)} \to (L^{(0)})^\perp$ such that

$$
\sup_{L^{(0)}} Dw_k^{(0)} \le 2 \sup_{\partial\Sigma_k^{(1)}} |\nu - \eta^{(0)}| \le 2 \operatorname*{osc}_{\partial\Sigma_k^{(1)}} \nu + 2 \operatorname*{osc}_{\partial\Sigma^{(0)}} \nu \le C\varepsilon,
$$

$$
f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x)) \in \partial\Sigma_k^{(1)} \quad \text{for } x \in \partial Q_k^{(1)},
$$

and

$$
f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x)) \in \Sigma_k^{(1)} \quad \text{for } x \in U_k^{(0)} \cap Q_k^{(1)},
$$

where $U_k^{(0)}$ is a neighborhood of $\partial Q_k^{(1)}$.

Recall that our goal is to prove that $\Sigma_1^{(1)}, \cdots, \Sigma_6^{(1)}$ are quasirectangles. Choose $\eta_k^{(1)}$ to be any value of ν on $\partial\Sigma_k^{(1)}$, let $L_k^{(1)}$ be the plane through the origin in $\mathbf{R}^n$ which is normal to the 2-vector $\eta_k^{(1)}$, $p_k^{(1)}$ the orthogonal projection onto $L_k^{(1)}$, and $q_k^{(1)}$ the orthogonal projection onto $(L_k^{(1)})^\perp$. Define $f_k^{(1)}: \mathbf{R}^2 \to L_k^{(1)}$ by

$$f_k^{(1)}(x) = p_k^{(1)}(f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x))).$$

Note that $\partial\Sigma_k^{(1)}$ projects simply onto $L_k^{(1)}$, that $f_k^{(1)}(\partial Q_k^{(1)}) = p_k^{(1)}(\partial\Sigma_k^{(1)}) = \Lambda_k^{(1)}$ and that

$$|\eta^{(0)} - \eta_k^{(1)}| \le \sup_{\partial\Sigma_k^{(1)}\cap\partial\Sigma^{(0)}} |\eta^{(0)} - \nu| + \sup_{\partial\Sigma_k^{(1)}\cap\partial\Sigma^{(0)}} |\eta_k^{(1)} - \nu| \le \operatorname*{osc}_{\partial\Sigma^{(0)}} \nu + \operatorname*{osc}_{\partial\Sigma_k^{(1)}} \nu.$$

By direct computation we have

$$df_k^{(1)} = p_k^{(1)}(df^{(0)} + d(w_k^{(0)} \circ f^{(0)})),$$

namely for $i, j = 1, 2$

$$\frac{\partial f_k^{(1)}}{\partial x_i} = \frac{\partial f^{(0)}}{\partial x_i} - q_k^{(1)}\left(\frac{\partial f^{(0)}}{\partial x_i}\right) + p_k^{(1)}\left(Dw_k^{(0)}\left(\frac{\partial f^{(0)}}{\partial x_i}\right)\right).$$

Since $q^{(0)}(\partial f^{(0)}/\partial x_i) = 0$ where $q^{(0)}$ is the orthogonal projection onto $(L^{(0)})^\perp$, and

$$p_k^{(1)}\left(Dw_k^{(0)}\left(\frac{\partial f^{(0)}}{\partial x_i}\right)\right) = (p_k^{(1)} - p^{(0)})\left(Dw_k^{(0)}\left(\frac{\partial f^{(0)}}{\partial x_i}\right)\right),$$

then

$$\begin{aligned}
&((df_k^{(1)})^* \circ (df_k^{(1)}))_{ij} - ((df^{(0)})^* \circ (df^{(0)}))_{ij} \\
&\quad = \left\langle (q^{(0)} - q_k^{(1)})\left(\frac{\partial f^{(0)}}{\partial x_i}\right), \left(q_k^{(1)} - q^{(0)}\right)\left(\frac{\partial f^{(0)}}{\partial x_j}\right)\right\rangle \\
&\qquad + 3\left\langle (p_k^{(1)} - p^{(0)})\left(Dw_k^{(0)}\left(\frac{\partial f^{(0)}}{\partial x_i}\right)\right), (p_k^{(1)} - p^{(0)})\left(\frac{\partial f^{(0)}}{\partial x_j}\right)\right\rangle.
\end{aligned}$$

Hence

$$\sup_{\mathbf{R}^2} |((df_k^{(1)})^* \circ (df_k^{(1)}))_{ij} - ((df^{(0)})^* \circ (df^{(0)}))_{ij}|$$
$$\leq (|\eta_k^{(1)} - \eta^{(0)}|^2 + 4 \sup|Dw_k^{(0)}|^2 + 4 \sup|Dw_k^{(0)}||\eta_k^{(1)} - \eta^{(0)}|)(\operatorname{Lip} f^{(0)})^2,$$

and since $\operatorname{Lip} f^{(0)} \leq \frac{9}{8}$ (see Remark 2.1 following Definition 2.1), applying Cauchy-Schwarz we have

$$\begin{aligned} &\sup_{\mathbf{R}^2} |((df_k^{(1)})^* \circ (df_k^{(1)}))_{ij} - ((df^{(0)})^* \circ (df^{(0)}))_{ij}| \\ &\quad \leq C(\eta_k^{(1)} - \eta^{(0)}|^2 + \sup|Dw_k^{(0)}|^2) \\ &\quad \leq C \left(\operatorname*{osc}_{\partial\Sigma^{(0)}}{}^2 \nu + \operatorname*{osc}_{\partial\Sigma_k^{(1)}}{}^2 \nu \right), \end{aligned}$$

which implies

$$\begin{aligned} \sup_{x\in\mathbf{R}^2} \|((df_k^{(1)})^* \circ (df_k^{(1)}))_x - \iota_x\| &\leq \sup_{x\in\mathbf{R}^2} \|((df^{(0)})^* \circ df^{(0)})_x - \iota_x\| \\ &\quad + C \left(\operatorname*{osc}_{\partial\Sigma^{(0)}}{}^2 \nu + \operatorname*{osc}_{\partial\Sigma_k^{(1)}}{}^2 \nu \right) \\ &\leq C_0 \varepsilon^2 . \end{aligned} \tag{5}$$

Remark 2.1 allows us to conclude that $f_k^{(1)}$ is a bilipschitz map from $\mathbf{R}^2$ onto $L_k^{(1)}$. This fact combined with the remark that $f_k^{(1)}(\partial Q_k^{(1)}) = p_k^{(1)}(\partial\Sigma_k^{(1)}) = \Lambda_k^{(1)}$ guarantees that $f_k^{(1)}(Q_k^{(1)}) = R_k^{(1)}$, where $R_k^{(1)}$ is the compact region of $L_k^{(1)}$ bounded by $\Lambda_k^{(1)}$.

In view of (4) and (5) we have shown that for each $k = 1, \cdots, 6$, $\Sigma_k^{(1)}$ is a quasirectangle (choosing ε small enough so that $C\varepsilon < \frac{1}{8}$), with $f_k^{(1)} : \mathbf{R}^2 \to L_k^{(1)}$, $Q_k^{(1)}$, $R_k^{(1)}$, $\eta_k^{(1)}$ corresponding to $f : \mathbf{R}^2 \to L, Q, R, \eta$ in the definition of quasirectangle.

By Remark 2.2 we know that there exist a smooth function $w_k^{(1)} : L_k^{(1)} \to (L_k^{(1)})^\perp$ and a neighborhood $W_k^{(1)}$ of $\partial R_k^{(1)} = \Lambda_k^{(1)} = f_k^{(1)}(\partial Q_k^{(1)})$ such that graph $w_k^{(1)}{}_{|W_k^{(1)}\cap R_k^{(1)}} = V_k^{(1)}$ for some boundary neighborhood $V_k^{(1)}$ of $\Sigma_k^{(1)}$, $\partial\Sigma_k^{(1)} = \operatorname{graph} w_k^{(1)}{}_{|\partial R_k^{(1)}}$, and

$$\sup_{L_k^{(1)}} |Dw_k^{(1)}| \leq \operatorname*{osc}_{L_k^{(1)}} Dw_k^{(1)} \leq 4 \operatorname*{osc}_{\partial\Sigma_k^{(1)}} \nu \leq C\varepsilon .$$

The previous construction shows that for $x \in \partial Q_k^{(1)}$

$$\begin{aligned} f_k^{(1)}(x) + w_k^{(1)}(f_k^{(1)}(x)) &= p_k^{(1)}(f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x))) \\ &\quad + w_k^{(1)}(f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x))) \\ &= p_k^{(1)}(f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x))) \\ &\quad + q_k^{(1)}(f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x))) \\ &= f^{(0)}(x) + w_k^{(0)}(f^{(0)}(x)), \end{aligned}$$

in particular, for $x \in \partial Q_k^{(1)} \cap \partial Q_q^{(1)}$

$$f_k^{(1)}(x) + w_k^{(1)}(f_k^{(1)}(x)) = f_q^{(1)}(x) + w_q^{(1)}(f_q^{(1)}(x)). \tag{6}$$

Part 3. We can repeat the slicing procedure, starting with $\Sigma_k^{(1)}$, $Q_k^{(1)}$, $f_k^{(1)}$, $L_k^{(1)}$ in place of $\Sigma^{(0)}$, $Q^{(0)}$, $f^{(0)}$, $L^{(0)}$ respectively, to generate $\Sigma_k^{(2)}$, $Q_k^{(2)}$, $f_k^{(2)}$, $L_k^{(2)}$, $1 \le k \le 6^2$. In fact we can repeat the slicing procedure j times, generating $\Sigma_k^{(l)}$, $Q_k^{(l)}$, $f_k^{(l)}$, $L_k^{(l)}$, $1 \le k \le 6^l$, for each $l = 1, \cdots, j$, provided

$$C_0^j \varepsilon^2 < (1/8)^2, \tag{7}$$

where C_0 is the constant appearing in the second inequality of (5). Indeed (according to the definition of α-quasirectangle) we only need to stop when we get to the first integer j such that

$$\sup_{\mathbf{R}^2} \|(df_k^{(j)})^* \circ (df_k^{(j)}) - \imath\| \ge (1/8)^2 \quad \text{or} \quad \operatorname*{osc}_{\partial\Sigma_k^{(j)}} \nu \ge 1/8 \tag{8}$$

for some $k \in \{1, \cdots, 6^j\}$. There is a useful criterion, more precise that (7) which guarantees that (8) cannot occur, provided that ε is chosen small enough to begin with. Namely, suppose that we have successfully iterated the slicing procedure j times, generating $\Sigma_k^{(l)}$, $Q_k^{(l)}$, $f_k^{(l)}$, $L_k^{(l)}$, $1 \le k \le 6^l$, for each $l = 1, \cdots, j$. Then a simple induction based on the first inequality in (5) shows that if $\{\Sigma_{k_l}^{(l)}\}_{l=0}^j$ with $k_l \in \{1, \cdots, 6^l\}$ is an arbitrary nested sequence of the $\Sigma_k^{(l)}$ (i.e., $\Sigma_{k_l}^{(l)} \subset \Sigma_{k_{l-1}}^{(l-1)}$ for $1 \le l \le j$), then

$$\sup_{\mathbf{R}^2} \|(df_k^{(j)})^* \circ (df_k^{(j)}) - \imath\| \le \sup_{\mathbf{R}^2} \|(df^{(0)})^* \circ (df^{(0)}) - \imath\| + C_1 \sum_{l=0}^{j} \operatorname*{osc}_{\partial\Sigma_{k_l}^{(l)}}{}^2 \nu \tag{9}$$

for suitable C_1 (in fact twice the constant C which appears in the first inequality of (5)). Thus we will be able to prove that (8) cannot occur if we can show that the sum $\sum_{l=0}^{j} \operatorname{osc}^2_{\partial\Sigma_{k_l}^{(l)}} \nu$ remains small (independent of

j) for any such sequence. It becomes clear now how crucial the choice of $\gamma_1, \gamma_2, \gamma_3$ made in Part 1 is.

Recall that we label the edges of $Q^{(0)}$ parallel to the x-axis $s_1(Q^{(0)})$, $s_3(Q^{(0)})$ and the edges parallel to the y-axis $s_2(Q^{(0)}), s_4(Q^{(0)})$ (with $s_1(Q^{(0)})$, $s_2(Q^{(0)})$ having smaller y and x coordinates than $s_3(Q^{(0)})$, $s_4(Q^{(0)})$ respectively). This labelling induces a corresponding labelling for the edges of $Q_k^{(1)}$, $\partial\Sigma^{(0)}$, and $\partial\Sigma_k^{(1)}$; in particular

$$\begin{aligned} s_i(\Sigma^{(0)}) &= (p^{(0)})^{-1}(f^{(0)}(s_i(Q^{(0)}))) \cap \partial\Sigma^{(0)}, \\ s_i(\Sigma_k^{(1)}) &= (p^{(0)})^{-1}(f^{(0)}(s_i(Q_k^{(1)}))) \cap \partial\Sigma_k^{(1)} \\ &= (p_k^{(1)})^{-1}(f_k^{(1)}(s_i(Q_k^{(1)}))) \cap \partial\Sigma_k^{(1)}, \end{aligned}$$

where the last equality comes from the fact that for $x \in \partial Q_k^{(1)}$

$$f_k^{(1)}(x) + w_k^{(1)}(f_k^{(1)}(x)) = f^{(0)}(x) + w^{(0)}(f^{(0)}(x)).$$

Notice that (2) gives

$$\mathscr{H}(s_i(\Sigma_k^{(1)})) \int_{s_i(\Sigma_k^{(1)})} |A|^2 \le C \min\left\{ \int_{S_{i,1}} |A|^2, \int_{S_{i,2}} |A|^2 \right\}, \tag{10}$$

provided that $s_i(\Sigma_k^{(1)})$ is one of the new edges of $\Sigma_k^{(1)}$, that is, $s_i(\Sigma_k^{(1)}) \not\subset s_i(\Sigma^{(0)})$; where $S_{i,1} = \{\zeta \in \Sigma^{(0)} : g_i(\zeta) \in I_{i,1}\}$ and $S_{i,2} = \{\zeta \in \Sigma^{(0)} : g_i(\zeta) \in I_{i,2}\}$. Note that by construction $\operatorname{dist}(I_{i,2}, I_{i,2}) = \frac{1}{30}|s_i(Q^{(0)})|$ and

$$\begin{aligned} \operatorname{dist}(S_{i,1}, S_{i,2}) &\ge (\operatorname{Lip} h^{(0)})^{-1} \operatorname{dist}(I_{i,1}, I_{i,2}) \ge \frac{8}{9} \cdot \frac{1}{30} |s_i(Q^{(0)})| \\ &= \frac{4}{135} |s_i(Q^{(0)})| \ge \frac{2}{75} \cdot \frac{\sqrt{5}}{2} \operatorname{diam} Q^{(0)}. \end{aligned} \tag{11}$$

Since for $1 \le l \le j$, $k_l \in \{1, \cdots, 6^l\}$, $k_{l-1} \in \{1, \cdots, 6^{l-1}\}$, and $Q_{k_l}^{(l)} \subset Q_{k_{l-1}}^{(l-1)}$, $\operatorname{diam} Q_{k_l}^{(l)} \le \frac{8}{15} \operatorname{diam} Q_{k_{l-1}}^{(l-1)}$, we have

$$\operatorname{diam} Q_k^{(j)} \le (8/15)^j \operatorname{diam} Q^{(0)}. \tag{12}$$

Combining Lemma 2.1, (11) and (12) we have

$$\operatorname{diam} \Sigma_k^{(j)} \le 24 \operatorname{diam} Q_k^{(j)} \le 24(8/15)^j \operatorname{diam} Q^{(0)},$$

and

$$\begin{aligned} \operatorname{diam} \Sigma_k^{(j)} \le 24 \left(\frac{8}{15}\right)^j \operatorname{diam} Q^{(0)} &\le \frac{\sqrt{5}}{75} \operatorname{diam} Q^{(0)} \\ &\le \operatorname{dist}(S_{i,1}, S_{i,2}) \quad \text{for } j \ge 11. \end{aligned}$$

Thus for $j \geq 11$, no $\Sigma_k^{(j)}$ can intersect both $S_{i,1}$ and $S_{i,2}$, and hence (10) implies

$$\mathcal{H}(s_i(\Sigma_k^{(1)})) \int_{s_i(\Sigma_k^{(1)})} |A|^2 \leq C \int_{\Sigma^{(0)} \setminus \Sigma_{k_{11}}^{(11)}} |A|^2 \quad \text{if } s_i(\Sigma_k^{(1)}) \not\subset s_i(\Sigma^{(0)}) \tag{13}$$

for each $1 \leq k \leq 6$. Now this can be applied with any of the iterates $\Sigma_k^{(j)}$ in place of $\Sigma^{(0)}$, so long as $\sigma^{(j+11)}$ is well defined; thus if $j \geq 1$ and $\Sigma_k^{(l)}$ are defined for $k \leq 6^l$ and each $l \leq j + 11$, and if $\Sigma_{k_l}^{(l)} \subset \cdots \subset \Sigma_{k_2}^{(2)} \subset \Sigma_{k_1}^{(1)} \subset \Sigma^{(0)}$ is any nested sequence, with $1 \leq k_l \leq 6^l$ for each l, then (13) actually implies

$$\mathcal{H}(s_i(\Sigma_{k_l}^{(l)})) \int_{s_i(\Sigma_{k_l}^{(l)})} |A|^2 \leq C \int_{\Sigma_{k_{l-1}}^{l-1} \setminus \Sigma_{k_{l+10}}^{(l+10)}} |A|^2 \quad \text{if } s_i(\Sigma_{k_l}^{(l)}) \not\subset s_i(\Sigma_{k_{l-1}}^{(l-1)}). \tag{14}$$

By the construction, for any nested sequence we have

$$\begin{aligned}
\mathcal{H}(s_i(\Sigma_{k_l}^{(l)})) &\leq \left(1 + \sup_{L_{k_l}^{(l)}} |Dw_{k_l}^{(l)}|^2\right) \operatorname{Lip} f_{k_l}^{(l)} |s_i(Q_{k_l}^{(l)})| \\
&\leq \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{8}{15} |s_i(Q_{k_{l-1}}^{(l-1)})| \leq \left(\frac{9}{8}\right)^2 \cdot \frac{8}{15} \cdot \frac{5}{4} \mathcal{H}(s_i(\Sigma_{k_{l-1}}^{(l-1)})) \\
&\leq \frac{27}{32} \mathcal{H}(s_i(\Sigma_{k_{l-1}}^{(l-1)})) \leq \frac{7}{8} \mathcal{H}(s_i(\Sigma_{k_{l-1}}^{(l-1)})),
\end{aligned}$$

for $l \leq j + 11$, where $w_{k_l}^{(l)}$ satisfies the same conditions as the function w constructed in Remark 2.2 did, but with respect to $L_{k_l}^{(l)}$, $\Sigma_{k_l}^{(l)}$ and $R_{k_l}^{(l)}$ in place of L, Σ, and R, respectively. Therefore

(15)

$$\mathcal{H}(s_i(\Sigma_{k_l}^{(l)})) \int_{s_i(\Sigma_{k_l}^{(l)})} |A|^2 \leq \frac{7}{8} \mathcal{H}(s_i(\Sigma_{k_{l-1}}^{(l-1)})) \int_{s_i(\Sigma_{k_{l-1}}^{(l-1)})} |A|^2$$

$$\text{if } s_i(\Sigma_{k_l}^{(l)}) \subset s_i(\Sigma_{k_{l-1}}^{(l-1)}).$$

Now consider the alternatives

(I) $\quad s_i(\Sigma_{k_j}^{(j)}) \subset \cdots \subset s_i(\Sigma_{k_{j-l+1}}^{(j-k+1)}) \not\subset s_i(\Sigma_{k_{j-l}}^{(j-l)})$ for some $1 \leq l \leq j$

or

(II) $\quad s_i(\Sigma_{k_j}^{(j)}) \subset \cdots \subset s_i(\Sigma^{(0)})$.

In case alternative (I) holds for some l, from (14) and (15) it follows that

$$\mathcal{H}(s_i(\Sigma_{k_j}^{(j)})) \int_{s_i(\Sigma_{k_j}^{(j)})} |A|^2 \leq C \left(\frac{7}{8}\right)^l \int_{\Sigma_{k_{j-l}}^{(j-l)} \setminus \Sigma_{k_{j-l+11}}^{(j-l+11)}} |A|^2,$$

while in the case where alternative (II) holds we deduce that

$$\mathscr{H}(s_i(\Sigma_{k_j}^{(j)}))\int_{s_i(\Sigma_{k_j}^{(j)})}|A|^2\le C\left(\frac{7}{8}\right)^j\mathscr{H}(s_i(\Sigma^{(0)}))\int_{s_i(\Sigma^{(0)})}|A|^2.$$

Since $\partial\Sigma_{k_j}^{(j)}$ is connected, we have

$$\operatorname*{osc}_{\partial\Sigma_{k_j}^{(j)}}\nu\le\sum_{i=1}^{4}\operatorname*{osc}_{s_i(\Sigma_{k_j}^{(j)})}\nu\quad\text{and}\quad\operatorname*{osc}_{\partial\Sigma_{k_j}^{(j)}}{}^2\nu\le 4\sum_{i=1}^{4}\operatorname*{osc}_{s_i(\Sigma_{k_j}^{(j)})}{}^2\nu.$$

Thus, regardless of which one of the alternatives (I), (II) holds, the following is true:

$$\operatorname*{osc}_{\partial\Sigma_{k_j}^{(j)}}{}^2\nu\le C\sum_{l=0}^{j}\left(\frac{7}{8}\right)^l\int_{\Sigma_{k_{j-l}}^{(j-l)}\setminus\Sigma_{k_{j-l+11}}^{(j-l+11)}}|A|^2+C\left(\frac{7}{8}\right)^j\mathscr{H}(\partial\Sigma^{(0)})\int_{\partial\Sigma^{(0)}}|A|^2.$$

Since this is also valid for any $q\le j$ in place of j, by summation we obtain that

$$\begin{aligned}\sum_{q=0}^{j}\operatorname*{osc}_{\partial\Sigma_{k_q}^{(q)}}{}^2\nu\le{}&\sum_{q=0}^{j}\sum_{l=0}^{q}\left(\frac{7}{8}\right)^l\int_{\Sigma_{k_{q-l}}^{(q-l)}\setminus\Sigma_{k_{q-l+11}}^{(q-l+11)}}|A|^2\\&+C\sum_{q=0}^{j}\left(\frac{7}{8}\right)^q\mathscr{H}(\partial\Sigma^{(0)})\int_{\partial\Sigma^{(0)}}|A|^2.\end{aligned}\tag{16}$$

But

$$\begin{aligned}&\sum_{q=0}^{j}\sum_{l=0}^{q}\left(\frac{7}{8}\right)^l\int_{\Sigma_{k_{q-l}}^{(q-l)}\setminus\Sigma_{k_{q-l+11}}^{(q-l+11)}}|A|^2\\&\quad=\sum_{q=0}^{j}\sum_{l=0}^{q}\left(\frac{7}{8}\right)^{q-l}\int_{\Sigma_{k_l}^{(l)}\setminus\Sigma_{k_{l+11}}^{(l+11)}}|A|^2=\sum_{l=0}^{j}\sum_{p=0}^{j-l}\left(\frac{7}{8}\right)^p\int_{\Sigma_{k_l}^{(l)}\setminus\Sigma_{k_{l+11}}^{(l+11)}}|A|^2\\&\quad\le 8\sum_{l=0}^{j}\int_{\Sigma_{k_l}^{(l)}\setminus\Sigma_{k_{l+11}}^{(l+11)}}|A|^2\le(8\times 11)\int_{\Sigma^{(0)}}|A|^2,\end{aligned}$$

and hence (16) implies

$$\sum_{q=0}^{j}\operatorname*{osc}_{\partial\Sigma_{k_q}^{(q)}}{}^2\nu\le C\left(\int_{\Sigma^{(0)}}|A|^2+\mathscr{H}(\partial\Sigma^{(0)})\int_{\partial\Sigma^{(0)}}|A|^2\right)\le C\varepsilon^2,\tag{17}$$

provided only that $\Sigma_k^{(j+11)}$ is well defined by the iterative slicing procedure described above. Note that we have used the additional hypothesis $\mathscr{H}(\partial\Sigma^{(0)})\int_{\partial\Sigma^{(0)}}|A|^2\le\varepsilon^2$.

Let $\beta_0 \in (0, \frac{1}{8})$ be an arbitrary value to be specified later and $\Sigma^{(0)}$ be an ε-quasirectangle with $\varepsilon \le \beta_0$. Suppose $j \ge 1$ is such that $\Sigma_k^{(j)}$ is well defined and

$$\sum_{l=0}^{j} \operatorname*{osc}_{\partial\Sigma_{k_l}^{(l)}}{}^2 \nu \le \beta_0^2$$

for every nested sequence $\{\Sigma_{k_l}^{(l)}\}_{l=0}^{j}$ with $k_l \in \{1, \cdots, 6^l\}$. In particular for $q \le j$, by inequality (9), we have

$$\sup_{\mathbf{R}^2} \|(df_{k_q}^{(q)})^* \circ (df_{k_q}^{(q)}) - \imath\| \le 2C_1\beta_0^2,$$

which implies that for any $q \le j$ and any $k_q \in \{1, \cdots, 6^q\}$, $\Sigma_{k_q}^{(q)}$ is a $C_2\beta_0$-quasirectangle, where $C_2 = \sqrt{2C_1}$. We then choose β_0 so that

$$C_0^{11}(C_2\beta_0)^2 < (1/8)^2;$$

in particular β_0 is independent of $\Sigma^{(0)}$ or j. This inequality asserts that condition (7) is satisfied with $C_2\beta_0$ in place ε. Thus $\Sigma_{k_l}^{(l)}$ is well defined for all $l \le j + 11$ and $k_l \in \{1, \cdots, 6^l\}$. Hence by (17) we have

$$\sum_{l=0}^{j} \operatorname*{osc}_{\partial\Sigma_{k_l}^{(l)}}{}^2 \nu \le C\varepsilon^2. \tag{18}$$

Also using (4), for any $\Sigma_{k_{j+1}}^{(j+1)} \subset \Sigma_{k_j}^{(j)}$ with $k_{j+1} \in \{1, \cdots, 6^{j+1}\}$ we obtain

$$\operatorname*{osc}_{\partial\Sigma_{k_{j+1}}^{(j+1)}} \nu \le C\left(\int_{\Sigma^{(0)}} |A|^2\right)^{1/2} + \operatorname*{osc}_{\partial\Sigma_{k_j}^{(j)}} \nu \le C\varepsilon. \tag{19}$$

Combining (18) and (19) we conclude that

$$\sum_{l=0}^{j+1} \operatorname*{osc}_{\partial\Sigma_{k_l}^{(l)}}{}^2 \nu \le C\varepsilon^2 \le \beta_0^2$$

for $\varepsilon \le \varepsilon_0$ where $\varepsilon_0 \in (0, \frac{1}{8})$ is a fixed constant not depending on $\Sigma^{(0)}$ or j and for every nested sequence $\{\Sigma_{k_l}^{(l)}\}_{l=0}^{j+1}$ with $k_l \in \{1, \cdots, 6^l\}$.

We have proved that there exists a fixed constant $\varepsilon_0 \in (0, \frac{1}{8})$ such that if $\varepsilon \in (0, \varepsilon_0]$, if $\Sigma^{(0)}$ satisfies the hypothesis of the main lemma for ε, and if the $\Sigma_k^{(j)}$ are well defined by the above slicing procedure with $j \ge 0$ and $\sum_{l=0}^{j} \operatorname{osc}^2_{\partial\Sigma_{k_l}^{(l)}} \nu \le \beta_0^2$ for every nested sequence $\{\Sigma_{k_l}^{(l)}\}_{l=0}^{j}$, then the $\Sigma_k^{(l)}$ are well defined for $l \le j+11$, $\sum_{l=0}^{j} \operatorname{osc}^2_{\partial\Sigma_{k_l}^{(l)}} \le C\varepsilon^2$ and $\sum_{l=0}^{j+1} \operatorname{osc}^2_{\partial\Sigma_{k_l}^{(l)}} \nu \le \beta_0^2$ for every nested sequence $\Sigma_{k_l}^{(l)}$, $l = 0, \cdots, j, j+1$. Thus by mathematical induction we can show that $\Sigma_k^{(j)}$ is well defined for any $j \ge 0$ and $k \in \{1, \cdots, 6^j\}$; moreover for all $j \ge 0$, $\sum_{l=0}^{j} \operatorname{osc}^2_{\partial\Sigma_{k_l}^{(l)}} \nu \le C\varepsilon^2$ for every nested sequence $\Sigma_{k_l}^{(l)}$.

Part 4. Another way of phrasing the above conclusions is as follows: there exists a fixed constant $\varepsilon_0 \in (0, \frac{1}{8})$ such that if $\varepsilon \in (0, \varepsilon_0]$, and $\Sigma^{(0)}$ satisfies the hypothesis of the Main Lemma for such ε, then for every $j \ge 0$, $\Sigma^{(0)}$ can be partitioned into 6^j ε_j-quasirectangles $\{\Sigma_k^{(j)}\}_{k=1}^{6^j}$, with $\varepsilon_j \le C\varepsilon$. In particular for every $j \ge 0$ and every $k = 1, \cdots, 6^j$ there is a plane $L_k^{(j)}$ orthogonal to the 2-vector $\eta_k^{(j)}$, a rectangle $Q_k^{(j)}$, a compact region $R_k^{(j)} \subset L_k^{(j)}$, and a smooth map $f_k^{(j)}: \mathbf{R}^2 \to L_k^{(j)}$ satisfying conditions (i), (ii), and (iii) of Definition 2.1. By Remark 2.2, we know that there is a smooth function $w_k^{(j)}: L_k^{(j)} \to (L_k^{(j)})^\perp$ and a neighborhood $W_k^{(j)}$ of $\partial R_k^{(j)} = \Lambda_k^{(j)} = f_k^{(j)}(\partial Q_k^{(j)})$ such that $\operatorname{graph} w_k^{(j)}{}_{|W_k^{(j)} \cap R_k^{(j)}} = V_k^{(j)}$ for some boundary neighborhood $V_k^{(j)}$ of $\Sigma_k^{(j)}$, $\partial\Sigma_k^{(j)} = \operatorname{graph} w_k^{(j)}{}_{|\partial R_k^{(j)}}$, and

$$\sup_{L_k^{(j)}} |Dw_k^{(j)}| \le \operatorname*{osc}_{L_k^{(j)}} Dw_k^{(j)} \le 4 \operatorname*{osc}_{\partial\Sigma_k^{(j)}} \nu \le C\varepsilon.$$

In view of (6), for $x \in \partial Q_k^{(j)} \cap \partial Q_q^{(j)}$, $k, q \in \{1, \cdots, 6^j\}$ we have

$$f_k^{(j)}(x) + w_k^{(j)}(f_k^{(j)}(x)) = f_q^{(j)}(x) + w_q^{(j)}(f_q^{(j)}(x)).$$

Note that since $\Sigma^{(0)}$ is C^∞ and $\operatorname{diam}\Sigma_k^{(j)} \le 24(\frac{8}{15})^j \operatorname{diam} Q^{(0)}$, for large enough j, we can select the $w_k^{(j)}$ to be so that $\operatorname{graph} w_k^{(j)}{}_{|R_k^{(j)}} = \Sigma_k^{(j)}$. From now on we fix one such j. We define maps $\varphi_k^{(j)}: Q_k^{(j)} \to \Sigma_k^{(j)}$ by

$$\varphi_k^{(j)}(x) = f_k^{(j)}(x) + w_k^{(j)}(f_k^{(j)}(x)).$$

One can easily check that for all $k = 1, \cdots, 6^j$,

$$\sup_{Q_k^{(j)}} \|(d\varphi_k^{(j)})^* \circ (d\varphi_k^{(j)}) - \imath\| \le C\varepsilon^2,$$

which by Remark 2.1 implies that the Jacobians of $\varphi_k^{(j)}$ and $(\varphi_k^{(j)})^{-1}$ are bounded by $1 + C\varepsilon^2$. The map $\Phi: Q^{(0)} \to \Sigma^{(0)}$ defined by

$$\Phi = \varphi_k^{(j)} \quad \text{on } Q_k^{(j)}$$

is a well-defined homeomorphism. Moreover, it is clear that

$$\|(d\Phi)^* \circ (d\Phi) - \imath\|_{L^\infty(Q^{(0)})} \leq C\varepsilon^2 .$$

In order to obtain the appropriate Lipschitz bounds for Φ and Φ^{-1}, let $x, y \in Q^{(0)}$ and let σ denote the segment joining them, then

$$\begin{aligned}|\Phi(x) - \Phi(y)| &\leq d(\Phi(x), \Phi(y)) \leq \mathscr{H}(\Phi(\sigma)) \\ &\leq \sum_{k=1}^{6^j} \mathscr{H}(\varphi_k^{(j)}(\sigma)) \leq (1 + C\varepsilon^2) \sum_{k=1}^{6^j} |\sigma \cap Q_k^{(j)}|,\end{aligned}$$

and

$$|\Phi(x) - \Phi(y)| \leq d(\Phi(x), \Phi(y)) \leq (1 + C\varepsilon^2)|x - y| .$$

This inequality shows that Φ is Lipschitz with respect to both the intrinsic and the Euclidean metrics. Let $\gamma \subset \Sigma^{(0)}$ be a smooth curve joining $\Phi(x)$ and $\Phi(y)$ and such that

$$\text{length}\, \gamma = \mathscr{H}(\gamma) \leq (1 + C\varepsilon^2) d(\Phi(x), \Phi(y)),$$

then

$$\begin{aligned}|x - y| &\leq \mathscr{H}(\Phi^{-1}(\gamma)) \leq \sum_{k=1}^{6^j} \mathscr{H}(\Phi^{-1}(\gamma) \cap Q_k^{(j)}) \\ &\leq \sum_{k=1}^{6^j} \mathscr{H}((\varphi_k^{(j)})^{-1}(\gamma)) \leq (1 + C\varepsilon^2) \sum_{k=1}^{6^j} \mathscr{H}(\gamma \cap \Sigma_k^{(j)}) \\ &\leq (1 + C\varepsilon^2)\mathscr{H}(\gamma) \leq (1 + C\varepsilon^2) d(\Phi(x), \Phi(y)) .\end{aligned}$$

This establishes that Φ is a bilipschitz map with respect to the intrinsic distance on $\Sigma^{(0)}$.

3. Bilipschitz parameterization in the smooth case

Simon proved (see [11], [12]) that for $\mathscr{S}$ an arbitrary smooth surface in $\mathbf{R}^n$ if $\int_{\mathscr{S} \cap B_\rho(0)} |A|^2 \, d\mathscr{H}^2$ is small enough, then $\mathscr{S} \cap B_{\rho/2}(0)$ is well approximated by graphs of functions with small Lipschitz constant. The existence of this special type of decomposition allows us to conclude that for $\int_{\mathscr{S} \cap B_\rho(0)} |A|^2 \, d\mathscr{H}^2$ small enough there exists a quasirectangle $\Sigma^{(0)}$ containing a neighborhood of the origin, and satisfying the hypothesis of the

Main Lemma from the previous section. Moreover we deduce that the Euclidean distance and the intrinsic distance are equivalent on $\Sigma^{(0)}$; this allows us to conclude, thanks to the Main Lemma, that $\Sigma^{(0)}$ admits a bilipschitz parameterization.

Lemma (Approximate Graphical Decomposition [12]**).** *For any $\beta > 0$, there exists $\varepsilon_0 = e_0(\beta, n) > 0$ (independent of $\mathscr{S}$, ρ) such that if $\varepsilon \in (0, \varepsilon_0]$, if $\partial\mathscr{S} \cap B_\rho(0) = \varnothing$, if $\mathscr{H}^2(\mathscr{S} \cap B_\rho(0)) \leq \beta\rho^2$, and if $\int_{\mathscr{S}\cap B_\rho(0)} |A|^2\, d\mathscr{H}^2 \leq \varepsilon^2$, then the following holds: There are pairwise disjoint sets $P_1, \cdots, P_N \subset \mathscr{S}$ with*

$$\sum_{j=1}^{N} \operatorname{diam} P_j \leq C\varepsilon^{1/2}\rho$$

and a set $S \subset (3\rho/4, \rho)$ with $L(S) \geq \rho/16$ such that if $\sigma \in S$ then $\partial B_\sigma(0)$ intersects $\mathscr{S}$ transversely, $\partial B_\sigma(0) \cap (\bigcup_j P_j) = \varnothing$ and

$$\mathscr{S} \cap B_\sigma(0) = \bigcup_{i=1}^{M} D_{\sigma,i},$$

where each $D_{\sigma,i}$ is topologically a disc so that $\operatorname{diam} D_{\sigma,i} \geq C^{-1}\sigma$. Moreover there exist functions $u_i \in C^\infty(\overline{\Omega}_i, L_i^\perp)$ with L_i a plane in $\mathbf{R}^n$, Ω_i a smooth bounded domain in L_i of the form $\Omega_i = \Omega_i^0 \setminus (\bigcup_k d_{i,k})$, where Ω_i^0 is simply connected and $d_{i,k}$ are pairwise disjoint closed discs in L_i which do not intersect $\partial\Omega_i^0$, with $\operatorname{graph} u_i$ connected, and

$$\sup_{\Omega_i} \rho^{-1}|u_i| + \sup_{\Omega_i} |Du_i| \leq C\varepsilon^{1/2(2n-3)},$$

$$\operatorname{graph} u_i \cap B_\sigma(0) \subset D_{\sigma,i},$$

and $D_{\sigma,i} \setminus \operatorname{graph} u_i$ is a union of a subcollection of the P_j, and each P_j is topologically a disc.

We claim that there exists $\varepsilon_1 > 0$ so that if $\mathscr{S}$ satisfies the hypothesis of the lemma above for $\varepsilon \leq \varepsilon_1 \leq \varepsilon_0$, and if $\mathscr{D}_\sigma$ is the connected component of $\mathscr{S} \cap B_\sigma(0)$ containing the origin, then there exists a function $u \in C^\infty(\overline{\Omega}, (L')^\perp)$ with L' a plane in $\mathbf{R}^n$ containing the origin, Ω a smooth bounded domain in L' of the form $\Omega = \Omega^0 \setminus (\bigcup_k d_k)$, where Ω^0 is simply connected and d_k are pairwise disjoint closed discs in L', with graph u connected, and $\sup_\Omega |Du| \leq C\varepsilon$, $\operatorname{graph} u \cap B_{\sigma/2}(0) \subset \mathscr{D}_\sigma \cap B_{\sigma/2}(0)$, and $\mathscr{D}_\sigma \cap B_{\sigma/2}(0) \setminus \operatorname{graph} u = \bigcup_{j=1}^{N_\sigma} P_j$, where the P_j are pairwise disjoint topological discs and $\sum_j \operatorname{diam} P_j \leq \frac{1}{64}\sigma$. Note that since $0 \in \mathscr{D}_\sigma \cap B_{\sigma/2}(0)$, $\sup_\Omega |Du| \leq C\varepsilon$ and $\sum_j \operatorname{diam} P_j \leq \frac{1}{64}\sigma$ imply $\sigma^{-1}\sup_\Omega |u| \leq \frac{1}{32}$ for ε_1 small enough.

Using the notation from the Approximate Graphical Decomposition Lemma and assuming $D_{\sigma,i} = \mathscr{D}_\sigma$ it is easy to prove, using basic calculus, that there exist a point $X_0 \in \Omega_i$ and a set $\Omega^{(i)} \subset \Omega_i$ with $|\Omega^{(1)}| \geq \frac{1}{4}|\Omega_i^0|$ such that for all $X \in \Omega^{(1)}$

$$|Du_i(X) - Du_i(X_0)| \leq \frac{C}{\sigma}\int_{\Omega_i} |D^2 u_i| \leq C\left(\int_{\Omega_i} |D^2 u_i|^2\right)^{1/2}.$$

Since $|Du| \leq \frac{1}{2}$, we have

$$\int_{\Omega_i} |D^2 u_i|^2 \leq C\int_{\mathscr{D}_\sigma} |A|^2,$$

which implies that for all $X \in \Omega^{(1)}$,

$$|Du_i(X) - Du_i(X_0)| \leq C\left(\int_{\mathscr{D}_\sigma} |A|^2\right)^{1/2} \leq C\varepsilon.$$

Therefore for $X \in \Omega^{(1)}$

$$|\nu(X + u_i(X)) - \nu(X_0 + u_i(X_0))| \leq 2|Du_i(X) - Du_i(X_0)| \leq C\varepsilon,$$

where $\nu(\zeta)$ denotes the 2-vector orthogonal to $T_\zeta\mathscr{S}$. Let $K > 0$ be an arbitrary constant to be specified later and let $\zeta_0 = X_0 + u_i(X_0)$. Sard's theorem and the co-area formula [1, 3.2.22] guarantee that there exists $t \in (K\varepsilon/2, K\varepsilon)$ such that the set $\Gamma = \{\zeta \in \mathscr{D}_\sigma : |\nu(\zeta_0) - \nu(\zeta)| = t\}$ is contained in the union of finitely many pairwise disjoint Jordan curves and Jordan arcs with endpoints in $\partial\mathscr{D}_\sigma$, and

$$\mathscr{H}(\Gamma \cap \mathscr{D}_\sigma) \leq \frac{C}{K\varepsilon}\int_{\mathscr{D}_\sigma} |A| \leq \frac{C}{K}\sigma.$$

Let

$$\mathscr{B} = \{\zeta \in \mathscr{D}_\sigma : |\nu(\zeta_0) - \nu(\zeta)| \geq t\},$$
$$\mathscr{G} = \{\zeta \in \mathscr{D}_\sigma : |\nu(\zeta_0) - \nu(\zeta)| < t\}.$$

Let L'' be the plane through the point X_0 with unit normal $\nu(\zeta_0)$, and let p'' denote the orthogonal projection onto L''. Let $D = B_{\sigma/2}(0) \cap L''$. Then the Poincaré inequality implies that

$$\min\{|\mathscr{C}_1|, |\mathscr{C}_2|\} \leq C\sigma\mathscr{H}(p''(\Gamma)) \leq C\sigma^2/K$$

for any disjoint open subsets $\mathscr{E}_1, \mathscr{E}_2 \subset D\backslash p''(\Gamma)$ such that $\partial\mathscr{E}_i \cap D \subset p''(\Gamma)$ (see [6]). Hence by choosing K large enough, independent of Σ and ρ, we can guarantee that there is a unique component $\mathscr{E} \subset D\backslash p''(\Gamma)$ such that

$$|D\backslash\mathscr{E}| \le C\sigma^2/K \quad \text{and} \quad p''(\mathscr{G}) \subset \mathscr{E}.$$

If $\mathscr{B} = \bigcup_i \mathscr{B}_i$ where the $\mathscr{B}_i$ are the connected components of $\mathscr{B}$, then (see Lemma 2 from [11]),

$$\operatorname{diam}\mathscr{B}_i \le C\left(\int_{\mathscr{B}_i} |A|\, d\mathscr{H}^2 + \mathscr{H}(\partial\mathscr{B}_i)\right)$$

and

$$\sum_i \operatorname{diam}\mathscr{B}_i \le C\left(\frac{1}{K} + \varepsilon\right)\sigma \le \frac{C}{K}\sigma \le \frac{1}{64}\sigma,$$

for K large enough. The rest of the proof is now straightforward.

Without loss of generality we may assume that L' is the x_1, x_2-plane. In order to prove that if $\int_{\mathscr{S}\cap B_\rho(0)} |A|^2\, d\mathscr{H}^2$ is small, then there exists a quasirectangle containing a neighborhood of the origin, we let $\mathscr{D}_\sigma \cap B_{\sigma/2}(0) = \mathscr{D}$ and consider the function $s\colon \mathscr{D} \to \mathbf{R}$ defined by $s(x_1, x_2, \cdots, x_n) = \frac{1}{2}(|x_1 - x_2| + |x_1 + x_2|)$. In particular, s is a Lipschitz function whose Jacobian is bounded above by 2. The co-area formula implies that there exists $\sigma_0 \in (\sigma/4, \sigma/2)$ so that $s^{-1}(\sigma_0) \cap \mathscr{D}$ does not intersect $\bigcup_j P_j$,

$$\mathscr{H}(s^{-1}(\sigma_0) \cap \mathscr{D}) \le (C/\sigma)\mathscr{H}^2(\mathscr{D}) \le C\beta\sigma, \tag{2}$$

and

$$\int_{s^{-1}(\sigma_0)\cap\mathscr{D}} |A|^2\, d\mathscr{H} \le \frac{C}{\sigma}\int_{\mathscr{D}} J^* s|A|^2 \le \frac{C}{\sigma}\int_{\mathscr{S}\cap B_\rho(0)} |A|^2\, d\mathscr{H}^2. \tag{3}$$

Let $Q^{(0)} = [-\sigma_0, \sigma_0] \times [-\sigma_0, \sigma_0]$ and $\Sigma^{(0)} = \mathscr{D} \cap (Q^{(0)} \times \mathbf{R}^{n-2})$. From the Approximate Graphical Decomposition Lemma and the fact that $s^{-1}(\sigma_0) \cap \bigcup_j P_j = \varnothing$, we conclude that $\Sigma^{(0)}$ is diffeomorphic to the unit square and that $\partial\Sigma^{(0)}$ projects simply onto L'. Moreover putting (2) and (3) together we have

$$\operatorname*{osc}_{\partial\Sigma^{(0)}}{}^2 \nu \le \mathscr{H}(\partial\Sigma^{(0)}) \int_{\partial\Sigma^{(0)}} |A|^2 \le C\int_{\mathscr{S}\cap B_\rho(0)} |A|^2\, d\mathscr{H}^2 \le C\varepsilon^2. \tag{4}$$

We conclude that condition (ii) from Definition 2.1 is satisfied for $\eta = \nu(\zeta^{(0)})$ for some $\zeta^{(0)} \in \partial\Sigma^{(0)}$ and L orthogonal to η. Let p be the orthogonal projection onto L. For $x \in L'$ we define $f(x) = p(x + v(x))$, where $v \in C^{\infty}(L', (L')^{\perp})$, $v_{|Q^{(0)}\cap\Omega} = u_{|Q^{(0)}\cap\Omega}$, and $\sup_{L'} |Dv| \leq 2 \sup_{\Omega} |Du| \leq C\varepsilon$. Note that f is smooth and that $f(Q^{(0)}) = R^{(0)}$ where $R^{(0)}$ is the compact region bounded by $p(\partial\Sigma^{(0)})$. By direct computation we show that

$$\sup_{\mathbf{R}^2} \|(df)^* \circ (df) - \imath\| \leq C\varepsilon^2 .$$

Thus we have

Lemma 3.1. *For any $\beta > 0$, there exists $\varepsilon_1 = \varepsilon_1(\beta, n) > 0$ so that if $\mathcal{S}$ satisfies the hypothesis of the Approximate Graphical Decomposition Lemma for $\varepsilon \leq \varepsilon_1 \leq \varepsilon_0$, then there exists a quasirectangle $\Sigma^{(0)}$ containing a neighborhood of the origin and satisfying $\mathcal{H}(\partial\Sigma^{(0)}) \int_{\partial\Sigma^{(0)}} |A|^2 \leq \varepsilon^2$.*

In order to prove that $\Sigma^{(0)}$ admits a bilipschitz parameterization, the only thing left to do is to check that the Euclidean metric and the intrinsic metric are equivalent in $\Sigma^{(0)}$.

Lemma 3.2. *For any $\beta > 0$, there exists $\varepsilon_2 = \varepsilon_2(\beta, n) > 0$ so that if $\mathcal{S}$ satisfies the hypothesis of the Approximate Graphical Decomposition Lemma for $\varepsilon \leq \varepsilon_2 \leq \varepsilon_1$, and $\mathcal{D}$ is the connected component of $\Sigma \cap B_{\sigma/2}(0)$ containing the origin, then for any $\zeta \in \mathcal{D}$*

$$|\zeta| \leq d(0, \zeta) \leq (1 + C\varepsilon^2)|\zeta|.$$

Remark. From the previous lemma we deduce that for any given $\zeta_1, \zeta_2 \in \mathcal{S}$ if either

$$\int_{\mathcal{S}\cap B_{2|\zeta_1-\zeta_2|}(\zeta_1)} |A|^2 \, d\mathcal{H}^2 \leq \varepsilon^2 \quad \text{or} \quad \int_{\mathcal{S}\cap B_{2|\zeta_1-\zeta_2|}(\zeta_2)} |A|^2 \, d\mathcal{H}^2 \leq \varepsilon^2 ,$$

and if in either case ζ_1 and ζ_2 are in the same connected component, then

$$|\zeta_1 - \zeta_2| \leq d(\zeta_1, \zeta_2) \leq (1 + C\varepsilon^2)|\zeta_1 - \zeta_2| .$$

Proof of Lemma 3.2. Let $\zeta' \in \mathcal{D}$ with $|\zeta'| = \rho'$. Then there exists $\sigma' \in (\frac{3}{2}\rho', 2\rho')$ so that

$$\mathcal{D} \cap B_{\sigma'}(0) = \mathcal{G} \cup \left(\bigcup_i P_i\right) \text{ with } \mathcal{G} \subset \operatorname{graph} u \text{ and } \sum_i \operatorname{diam} P_i \leq \frac{1}{64}\sigma' ,$$

with $u \in C^{\infty}(\Omega, L^{\perp})$, $\Omega \subset L$, and L a plane containing the origin, and $\sup_{\Omega} |Du| \leq C\varepsilon$, and $(\sigma')^{-1} \sup|u| \leq \frac{1}{32}$. Let $\mathcal{F}(x) = x + u(x)$ for $x \in \Omega$ and let π denote the orthogonal projection onto L. The segment joining the origin to $\pi(\zeta')$ is the union of segments $[q_i, p_{i+1}]$ which are completely contained in Ω and segments $[p_i, q_i]$ which do not intersect Ω. We denote by $\kappa|\pi(\zeta')|$ the total length of the segments $[p_i, q_i]$, i.e., $\sum_i |p_i - q_i| = \kappa|\pi(\zeta')| \le \kappa|\zeta'|$. Note that $\kappa \le \frac{1}{16}$. Using this notation we have

$$\begin{aligned} d(0, \zeta') &\le \sum_{[p_i, q_i]\cap\Omega=\varnothing} d(\mathscr{F}(p_i), \mathscr{F}(q_i)) + \sum_{[q_i, p_{i+1}]\subset\Omega} d(\mathscr{F}(q_i), \mathscr{F}(p_{i+1})) \\ &\le (1 + C\varepsilon^2) \sum_{[q_i, p_{i+1}]\subset\Omega} |q_i - p_{i+1}| + \sum_{[p_i, q_i]\cap\Omega=\varnothing} d(\mathscr{F}(p_i), \mathscr{F}(q_i)) \\ &\le (1 + C\varepsilon^2)(1 - \kappa)|\zeta'| + \sum_i d(\mathscr{F}(p_i), \mathscr{F}(q_i)). \end{aligned}$$

Let $\zeta_i = \mathscr{F}(p_i)$ and $\eta_i = \mathscr{F}(q_i)$; then

$$\begin{aligned} \sum_i |\zeta_i - \eta_i| &= \sum_i (|p_i - q_i|^2 + |u(p_i) - u(q_i)|^2)^{1/2} \le (1 + C\varepsilon^2) \sum_i |p_i - q_i| \\ &\le \frac{1}{16}(1 + C\varepsilon^2)|\zeta'|. \end{aligned}$$

In order to evaluate $d(\zeta_i, \eta_i)$ we repeat the previous process replacing $0, \zeta', \mathscr{F}, \pi, p_i, q_i$, and κ by $\zeta_i, \eta_i, \mathscr{F}_i, \pi_i, p_{i,j}, q_{i,j}$, and κ_i, respectively. Then

$$d(\zeta_i, \eta_i) \le (1 + C\varepsilon^2)(1 - \kappa_i)|\zeta_i - \eta_i| + \sum_j d(\mathscr{F}_i(p_{i,j}), \mathscr{F}_i(q_{i,j})),$$

and

$$\begin{aligned} d(0, \zeta') \le (1 + C\varepsilon^2)(1 - \kappa)|\zeta'| &+ (1 + C\varepsilon^2)\sum_i (1 - \kappa_i)|\zeta_i - \eta_i| \\ &+ \sum_{i,j} d(\mathscr{F}_i(p_{i,j}), \mathscr{F}_i(q_{i,j})). \end{aligned}$$

Iterating the process k times and using the notation $\zeta_{i_1, \cdots, i_n}$, $\eta_{i_1, \cdots, i_n}$, $\mathscr{F}_{i_1, \cdots, i_n}$, $\pi_{i_1, \cdots, i_n}$, $p_{i_1, \cdots, i_{n+1}}$, $q_{i_1, \cdots, i_{n+1}}$, and $\kappa_{i_1, \cdots, i_n}$ in place of ζ_i, η_i, $\mathscr{F}_i, \pi_i, p_{i,j}, q_{i,j}$, and κ_i, respectively, for $n \le k$, we obtain

$$\begin{aligned} d(0, \zeta') \le\ & (1 + C\varepsilon^2)(1 - \kappa)|\zeta'| \\ & + (1 + C\varepsilon^2)\sum_{n=1}^{k} \sum_{i_1, \cdots, i_n} (1 - \kappa_{i_1, \cdots, i_n})|\zeta_{i_1, \cdots, i_n} - \eta_{i_1, \cdots, i_n}| \\ & + \sum_{i_1, \cdots, i_{k+1}} d(\mathscr{F}_{i_1, \cdots, i_k}(p_{i_1, \cdots, i_{k+1}}), \mathscr{F}_{i_1, \cdots, i_k}(q_{i_1, \cdots, i_{k+1}}), \end{aligned}$$

where

$$\begin{aligned}
\sum_{i_1,\cdots,i_{k+1}} &|\zeta_{i_1,\cdots,i_{k+1}} - \eta_{i_1,\cdots,i_{k+1}}| \\
&= \sum_{i_1,\cdots,i_{k+1}} |\mathscr{F}_{i_1,\cdots,i_k}(p_{i_1,\cdots,i_{k+1}}) - \mathscr{F}_{i_1,\cdots,i_k}(q_{i_1,\cdots,i_{k+1}})| \\
&\le (1+C\varepsilon^2) \sum_{i_1,\cdots,i_{k+1}} |p_{i_1}, \cdots, i_{k+1} - q_{i_1}, \cdots, i_{k+1}| \\
&\le (1+C\varepsilon^2) \sum_{i_1,\cdots,i_k} \kappa_{i_1,\cdots,i_k} |\zeta_{i_1,\cdots,i_k} - \eta_{i_1,\cdots,i_k}| \\
&\le \frac{1}{16}(1+C\varepsilon^2) \sum_{i_1,\cdots,i_k} |\zeta_{i_1,\cdots,i_k} - \eta_{i_1,\cdots,i_k}| \\
&\le \left(\frac{1}{16}(1+C\varepsilon^2)\right)^{k+1} |\zeta'|.
\end{aligned}$$

We choose $\varepsilon > 0$ so that $\lambda = \frac{1}{16}(1+C\varepsilon^2) \le \frac{1}{2}$. Since $\mathscr{D}$ is a compact smooth surface, we conclude that there exists $k > 0$ so that for all $i_1, \cdots, i_k, i_{k+1}$,

$$\begin{aligned}
d(\mathscr{F}_{i_1,\cdots,i_k}&(p_{i_1,\cdots,i_{k+1}}), \mathscr{F}_{i_1,\cdots,i_k}(q_{i_1,\cdots,i_{k+1}})) \\
&\le (1+\varepsilon^2)|p_{i_1,\cdots,i_{k+1}} - q_{i_1,\cdots,i_{k+1}}| \\
&\le (1+\varepsilon^2)|\zeta_{i_1,\cdots,i_{k+1}} - \eta_{i_1,\cdots,i_{k+1}}|.
\end{aligned}$$

Under this assumption we have

$$\begin{aligned}
d(0,\zeta') \le &(1+C\varepsilon^2)(1-\kappa)|\zeta'| \\
&+ (1+C\varepsilon^2)\sum_{n=1}^{k} \sum_{i_1,\cdots,i_n} (1-\kappa_{i_1,\cdots,i_n})|\zeta_{i_1,\cdots,i_n} - \eta_{i_1,\cdots,i_n}| \\
&+ (1+C\varepsilon^2) \sum_{i_1,\cdots,i_{k+1}} |\zeta_{i_1,\cdots,i_{k+1}} - \eta_{i_1,\cdots,i_{k+1}}|,
\end{aligned}$$

$$\begin{aligned}
d(0,\zeta') \le &(1+C\varepsilon^2)(1-\kappa)|\zeta'| \\
&+ (1+C\varepsilon^2)\sum_{n=1}^{k} \sum_{i_1,\cdots,i_n} (1-\kappa_{i_1,\cdots,i_n})|\zeta_{i_1,\cdots,i_n} - \eta_{i_1,\cdots,i_n}| \\
&+ (1+C\varepsilon^2)^2 \sum_{i_1,\cdots,i_k} \kappa_{i_1,\cdots,i_k}|\zeta_{i_1,\cdots,i_k} - \eta_{i_1,\cdots,i_k}|,
\end{aligned}$$

$$d(0,\zeta') \le (1+C\varepsilon^2)(1-\kappa)|\zeta'| + (1+C\varepsilon^2)\sum_{n=1}^{k-1}\sum_{i_1,\cdots,i_n}(1-\kappa_{i_1,\cdots,i_n})|\zeta_{i_1,\cdots,i_n}-\eta_{i_1,\cdots,i_n}| + (1+C\varepsilon^2)\sum_{i_1,\cdots,i_k}|\zeta_{i_1,\cdots,i_k}-\eta_{i_1,\cdots,i_k}| + C\varepsilon^2(1+C\varepsilon^2)\sum_{i_1,\cdots,i_k}\kappa_{i_1,\cdots,i_k}|\zeta_{i_1,\cdots,i_k}-\eta_{i_1,\cdots,i_k}|,$$

$$d(0,\zeta') \le (1+C\varepsilon^2)(1-\kappa)|\zeta'| + (1+C\varepsilon^2)\sum_{n=1}^{k-1}\sum_{i_1,\cdots,i_n}(1-\kappa_{i_1,\cdots,i_n})|\zeta_{i_1,\cdots,i_n}-\eta_{i_1,\cdots,i_n}| + (1+C\varepsilon^2)\sum_{i_1,\cdots,i_k}|\zeta_{i_1,\cdots,i_k}-\eta_{i_1,\cdots,i_k}| + C\varepsilon^2\lambda^{k+1}|\zeta'|.$$

Therefore we conclude

$$d(0,\zeta') \le (1+C\varepsilon^2)|\zeta'| + C\varepsilon^2\left(\sum_{n=1}^{k+1}\lambda^n\right)|\zeta'| \le (1+C\varepsilon^2)|\zeta'|.$$

Since the inequality $|\zeta'| \le d(0,\zeta')$ holds for any $\zeta' \in \mathscr{S}$, the proof of Lemma 3.2 is complete.

Hence combining the results from the Main Lemma and Lemmas 3.1 and 3.2 we deduce

Theorem 3.1. *For any $\beta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\beta, n) > 0$ so that if $\mathscr{S}$ is a smooth surface in $\mathbf{R}^n$, if $\varepsilon \in (0, \varepsilon_0]$, if $\partial\mathscr{S} \cap B_\rho(0) = \varnothing$, if $\mathscr{H}^2(\mathscr{S} \cap B_\rho(0)) \le \beta\rho^2$, and if $\int_{\mathscr{S}\cap B_\rho(0)} |A|^2\, d\mathscr{H}^2 \le \varepsilon^2$, then*

$$\mathscr{S} \cap B_{\rho/16}(0) = \bigcup_i^N \mathscr{D}_i \cap B_{\rho/16}(0) \quad \textit{and} \quad \partial\mathscr{D}_i \cap B_{\rho/16}(0) = \varnothing,$$

where each $\mathscr{D}_i$ is the image of a disc in $\mathbf{R}^2$ by a bilipschitz map Φ_i. Moreover for $i = 1, \cdots, N$,

$$\|(d\Phi_i)^* \circ (d\Phi_i) - \imath\|_{L^\infty} \le C\varepsilon^2 \quad \textit{and} \quad \mathrm{Lip}\,\Phi_i,\ \mathrm{Lip}\,\Phi_i^{-1} \le 1 + C\varepsilon^2.$$

4. Bilipschitz parameterization in the varifold case

Recall that given $\beta, \varepsilon, \rho > 0$, we denote by $\mathscr{T}_{\beta,\varepsilon}(B_\rho(0))$ the set of C^∞ embedded connected surfaces $\mathscr{S}$ in $\mathbf{R}^n$, containing o with $\partial\mathscr{S} \cap B_\rho(0) = \varnothing$, and satisfying

$$\mathscr{H}^2(\mathscr{S} \cap B_\rho(0)) \le \beta\rho^2 \quad \text{and} \quad \int_{\mathscr{S}\cap B_\rho(0)} |A|^2\, d\mathscr{H}^2 \le \varepsilon^2,$$

where A denotes the second fundamental form of $\mathscr{S}$. We denote by $\overline{\mathscr{T}_{\beta,\varepsilon}(B_\rho(0))}$ the set of integer multiplicity varifolds $\underline{v}(\mathscr{S}, \theta)$ which in $B_\rho(0)$ can be expressed as a measure theoretic limit of sequences $\{\mathscr{S}_k\}$, where $\mathscr{S}_k \in \mathscr{T}_{\beta,\varepsilon}(B_\rho(0))$, i.e., $\int_{\mathscr{S}_k} f\, d\mathscr{H}^2 \to \int_{\mathscr{S}} f\, d\mu$ for each continuous function $f: B_\rho(0) \to \mathbf{R}$ with compact support in $B_\rho(0)$. In particular $\mu = \mathscr{H}^2 \llcorner \theta$. In order to prove the local result we study the class $\overline{\mathscr{T}_{\beta,\varepsilon}(B_\rho(0))}$, but other than simplifying the notation there is nothing special about the choice of $B_\rho(0)$ over $B_\rho(\zeta)$.

Lemma 4.1. *Let* $\{\mathscr{S}_k\} \subset \mathscr{T}_{\beta,\varepsilon}(B_\rho(0))$ *converge to* $\underline{v}(\mathscr{S}, \theta) \in \overline{\mathscr{T}_{\beta,\varepsilon}(B_\rho(0))}$ *in the above measure theoretic sense. Then* $\{\mathscr{S}_k\}$ *converges to* $\mathscr{S}$ *in the Hausdorff distance sense.*

Remark. It follows from the proof to be given below, that $\mathscr{S}$ has generalized second fundamental form A in the sense of [3] and that A is in L^2.

Proof. Let $\{\mathscr{S}_k\} \subset \mathscr{T}_{\beta,\varepsilon}(B_\rho(0))$ be so that $\int_{\mathscr{S}_k} f\, d\mathscr{H}^2 \to \int_{\mathscr{S}} f\, d\mu$ for any continuous function f with compact support in $B_\rho(0)$. By the monotonicity formula, for all $\zeta \in B_{\rho/2}(0)$ and for almost all $\tau \in (0, \rho/4)$,

$$\frac{d}{d\tau}(\tau^{-2}\mathscr{H}^2(\mathscr{S}_k \cap B_\tau(\zeta)) = \frac{d}{d\tau}\int_{\mathscr{S}_k\cap B_\tau(\zeta)} \frac{|D^\perp r|^2}{r^2}\, d\mathscr{H}^2 + \tau^{-3}\int_{\mathscr{S}_k \cap B_\tau(\zeta)} \langle (x-\zeta), \underline{H_k}\rangle\, d\mathscr{H}^2,$$

in the distribution sense, where $r = |x - \zeta|$. Integrating between σ and τ with $\sigma < \tau$, we have

$$\begin{aligned}
&\tau^{-2}\mathscr{H}^2(\mathscr{S}_k \cap B_\tau(\zeta)) - \sigma^{-2}\mathscr{H}^2(\mathscr{S}_k \cap B_\sigma(\zeta)) \\
&= \frac{-\tau^{-2}}{2}\int_{\mathscr{S}_k\cap B_\tau(\zeta)} \langle (x-\zeta), \underline{H_k}\rangle\, d\mathscr{H}^2 + \frac{\sigma^{-2}}{2}\int_{\mathscr{S}_k\cap B_\sigma(\zeta)} \langle (x-\zeta), \underline{H_k}\rangle\, d\mathscr{H}^2 \\
&+ \int_{(B_\tau(\zeta)\setminus B_\sigma(\zeta))\cap\mathscr{S}_k} \left(\frac{1}{4}\underline{H_k} + \frac{D^\perp r}{r}\right)^2 d\mathscr{H}^2 \\
&- \frac{1}{16}\int_{(B_\tau(\zeta)\setminus B_\sigma(\zeta))\cap\mathscr{S}_k} \left(\frac{1}{4}\underline{H_k} + \frac{D^\perp r}{r}\right)^2 d\mathscr{H}^2 \\
&- \frac{1}{16}\int_{(B_\tau(\zeta)\setminus B_\sigma(\zeta))\cap\mathscr{S}_k} |\underline{H_k}|^2\, d\mathscr{H}^2.
\end{aligned}$$

Applying Cauchy's inequality $(ab \le \varepsilon a^2/2 + b^2/2\varepsilon)$, and letting $\sigma \downarrow 0$ for $\zeta \in \mathcal{S}_k$ we deduce

$$(1) \qquad 2\tau^{-2}\mathscr{H}^2(\mathcal{S}_k \cap B_\tau(\zeta)) + \frac{1}{8}\int_{\mathcal{S}_k \cap B_\tau(\zeta)} |\underline{H_k}|^2 \, d\mathscr{H}^2 \ge \frac{\pi}{2}.$$

If $\mathcal{S}_k$ did not converge to $\mathcal{S}$ in the Hausdorff distance sense, then there would be a sequence $\{\eta_k\}$ with $\eta_k \in \mathcal{S}_k$, $\eta_k \to \eta$, and $\eta \notin \mathcal{S}$. Let $\tau > 0$ satisfy $\mathcal{S} \cap B_{2\tau}(\eta) = \varnothing$, and let k be large enough so that $|\eta_k - \eta| \le \tau$. Fix $N > 0$ and let $Nr = \tau$. Since $\mathcal{S}_k$ is connected and the $\mathcal{S}_k$ converge to $\mathcal{S}$, there exist points $p_1, \cdots, p_N \in \mathcal{S}_k$ such that $B_{r/4}(p_i) \subset B_{ir}(\eta_k) \backslash B_{(i-1)r}(\eta_k)$. Applying (1) with p_i in place of ζ, $r/4$ in place of τ, summing over i, and using the fact that $\bigcup_i^N B_{r/4}(p_i) \subset B_\tau(\eta_k) \subset B_{2\tau}(\eta)$, we obtain

$$16\left(\frac{r}{4}\right)^{-2}\mathscr{H}^2(\mathcal{S}_k \cap B_{2\tau}(\eta)) + \int_{\mathcal{S}_k \cap B_{2\tau}(\eta)} |\underline{H_k}|^2 \, d\mathscr{H}^2 \ge 4N\pi.$$

Hence for all $N > 0$,

$$\liminf_{k\to\infty} \int_{\mathcal{S}_k \cap B_\rho(0)} |A_k|^2 \, d\mathscr{H}^2 \ge \liminf_{k\to\infty} \int_{\mathcal{S}_k \cap B_{2\tau}(\eta)} |A_k|^2 \, d\mathscr{H}^2 \ge 2N\pi,$$

where A_k denotes the second fundamental form of $\mathcal{S}_k$. This last inequality contradicts the fact that the L^2 norms of the A_k are uniformly bounded on $B_\rho(0)$.

Theorem. *For any $\beta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\beta, n)$ so that if $\underline{v}(\mathcal{S}, \theta) \in \overline{\mathcal{T}_{\beta,\varepsilon_0}(B_\rho(0))}$ and $0 \in \mathcal{S}$, then*

$$\underline{v}(\mathcal{S} \llcorner B_{\rho/64}(0)) = \sum_{i=1}^{N_0} \underline{v}(\mathcal{D}_i \llcorner B_{\rho/64}(0)),$$

where each $\mathcal{D}_i$ is the image of a disc in $\mathbf{R}^2$ via a bilipschitz map Φ_i, and where the decomposition is compatible with the multiplicity. Moreover for $i = 1, \cdots, N_0$,

$$\|(d\Phi_i)^* \circ (d\Phi_i) - \imath\|_{L^\infty} \le C\varepsilon_0^2 \quad \textit{and} \quad \operatorname{Lip}\Phi_i, \operatorname{Lip}\Phi_i^{-1} \le 1 + C\varepsilon_0^2.$$

Proof. For $\beta > 0$, let ε_0 be as in Theorem 3.1. Since $\underline{v}(\mathcal{S}, \theta) \in \overline{\mathcal{T}_{\beta,\varepsilon_0}(B_\rho(0))}$, there exists a sequence $\{\mathcal{S}_k\} \subset \mathcal{T}_{\beta,\varepsilon_0}(B_\rho(0))$ which converges to $\underline{v}(\mathcal{S}, \theta)$ in the measure theoretic sense on $B_\rho(0)$. In particular $\liminf_{k\to\infty} \int_{\mathcal{S}_k \cap B_\rho(0)} |A_k|^2 \, d\mathscr{H}^2 \le \varepsilon_0^2$, and $\{\mathcal{S}_k\}$ converges to $\mathcal{S}$ in the Hausdorff distance sense. Thus there exists a sequence $\{\zeta_k\}$ with $\zeta_k \in \mathcal{S}_k$

such that $\zeta_k \to 0$. For all k large enough, $B_{\rho/2}(\zeta_k) \subset B_\rho(0)$ and Theorem 3.1 guarantees

$$\mathcal{S}_k \cap B_{\rho/32}(\zeta_k) = \bigcup_{j=1}^{M_k} \Phi_j^k(D) \cap B_{\rho/32}(\zeta_k) \quad \text{and} \quad \partial\Phi_j^k(D) \cap B_{\rho/32}(\zeta_k) = \varnothing,$$

where Φ_j^k is a bilipschitz map from a disc $D \subset \mathbf{R}^2$ centered at the origin, onto one of the connected components of $\mathcal{S}_k \cap B_{\rho/32}(\zeta_k)$. Since the areas of the $\mathcal{S}_k$ are locally uniformly bounded, the M_k's are uniformly bounded independently of k (i.e., $\sup_k M_k \leq M$). By passing to a subsequence we may assume that $M_k = N \geq 1$ for all k, and that

$$\liminf_{k\to\infty} \int_{\mathcal{S}_k \cap B_\rho(0)} |A_k|^2 \, d\mathcal{H}^2 = \lim_{k\to\infty} \int_{\mathcal{S}_k \cap B_\rho(0)} |A_k|^2 \, d\mathcal{H}^2 .$$

Furthermore for each $j = 1, \cdots, N$,

$$\begin{aligned} \operatorname{Lip}\Phi_j^k, \operatorname{Lip}(\Phi_j^k)^{-1} &\leq 1 + C \int_{\mathcal{S}_k \cap B_{\rho/2}(\zeta_k)} |A_k|^2 \, d\mathcal{H}^2 \\ &\leq 1 + C \int_{\mathcal{S}_k \cap B_\rho(0)} |A_k|^2 \, d\mathcal{H}^2 , \end{aligned}$$

by Theorem 3.1. For fixed $j = 1, \cdots, N$, and Φ_j^k are equicontinuous and uniformly bounded. Thus by Arzela-Ascoli we conclude that there is a subsequence $\{\Phi_j^{k'}\}$ which converges uniformly to a bilipschitz map Φ_j. Without loss of generality we can choose a subsequence of $\{k'\}$ that works for all j. Since the $\mathcal{S}_{k'}$ converge to $\mathcal{S}$ in the Hausdorff distance sense,

$$\underline{v}(\mathcal{S} \llcorner B_{\sigma/64}(0)) = \sum_{j=1}^{N} \underline{v}(\Phi_j(D) \llcorner B_{\sigma/64}(0)), \quad \text{with } \partial\Phi_j(D) \cap B_{\sigma/64}(0) = \varnothing$$

and

$$\operatorname{Lip}\Phi_j, \operatorname{Lip}(\Phi_j)^{-1} \leq 1 + C \liminf_{k\to\infty} \int_{\mathcal{S}_k \cap B_\rho(0)} |A_k|^2 \, d\mathcal{H}^2 \leq 1 + C\varepsilon_0^2 .$$

Furthermore since for each $j = 1, \cdots, N$,

$$\|(d\Phi_j^k)^* \circ (d\Phi_j^k) - \imath\|_{L^\infty(D)} \leq C\varepsilon_0^2 ,$$

letting $k \to \infty$ we obtain

$$\|(d\Phi_j)^* \circ (d\Phi_j) - \imath\|_{L^\infty(D)} \leq C\varepsilon_0^2 .$$

Notice also that if $\zeta \in \mathcal{S} \cap B_{\sigma/64}(0)$, then $\theta(\zeta) = |\{j \in \{1, \cdots, N\} : \zeta \in \Phi_j(D) \cap B_{\sigma/64}(0)\}|$.

Recall that for an open domain $U \subset \mathbf{R}^n$ with $0 \in U$, $\mathcal{T}(U)$ denotes the set of multiplicity one 2-dimensional varifolds without boundary, $\underline{v}(\mathcal{S})$, with C^∞ connected support in U, containing 0 and which are have uniform local bounds in U on their areas and on the L^2 norms of their second fundamental form. We denote by $\overline{\mathcal{T}(U)}$ the set of $\underline{v}(\mathcal{S}, \theta)$ which in U, can be expressed as the measure theoretic limit of sequences $\{\underline{v}(\mathcal{S}_k)\}$, where $\underline{v}(\mathcal{S}_k) \in \mathcal{T}(U)$. That is, we assume that for each compact $K \subset U$ there is a constant C_K such that $\mathcal{H}^2(\mathcal{S}_k \cap K) \le C_K$, $\int_{\mathcal{S}_k \cap K} |A_k|^2 \, d\mathcal{H}^2 \le C_K$ and $\int_{\mathcal{S}_k} f \, d\mathcal{H}^2 \to \int_{\mathcal{S}} f \, d\mu$ for each fixed continuous $f : U \to \mathbf{R}$ with compact support in U. Under these conditions $\mu = \mathcal{H}^2 \llcorner \theta$, where θ is a positive integer valued function, and $\{\mathcal{S}_k\}$ converges to $\mathcal{S}$ in the Hausdorff distance sense.

Definition. Let $\underline{v}(\mathcal{S}, \theta) \in \overline{\mathcal{T}(U)}$, let $\varepsilon > 0$ we say that $\zeta \in \mathcal{S}$ is a bad point for ε if for every sequence $\{\underline{v}(\mathcal{S}_k)\} \subset \mathcal{T}(U)$ converging to $\underline{v}(\mathcal{S}, \theta)$ in the measure theoretic sense,

$$\lim_{\sigma \to 0} \left(\liminf_{k \to \infty} \int_{\mathcal{S}_k \cap B_\sigma(\zeta)} |A_k|^2 \, d\mathcal{H}^2 \right) > \varepsilon^2 .$$

Note that for a given $\varepsilon > 0$ there are finitely many bad points $\zeta_1, \cdots, \zeta_p$, with $p = p(\varepsilon)$. If $\zeta \in \mathcal{S} \setminus \{\zeta_1, \cdots, \zeta_p\}$, then we say that ζ is a good point for ε.

Corollary 4.1. *There exists $\varepsilon_0 > 0$ so that if $\varepsilon \in (0, \varepsilon_0]$, if $\underline{v}(\mathcal{S}, \theta) \in \overline{\mathcal{T}(U)}$, and if $\zeta \in \mathcal{S}$ is a good point for ε, then there exists $r(\zeta) > 0$ such that for all $0 < r \le r(\zeta)$*

$$\underline{v}(\mathcal{S} \llcorner B_r(\zeta)) = \sum_{i}^{N_\zeta} \underline{v}(\mathcal{D}_i \angle B_r(\zeta)),$$

where each $\mathcal{D}_i$ is a bilipschitz image of a disc in $\mathbf{R}^2$, and the decomposition is compatible with the multiplicity. Thus if $\zeta' \in \mathcal{S} \cap B_r(\zeta)$ has multiplicity l, then precisely l of these discs $\mathcal{D}_i$ contain ζ'.

Corollary 4.2. *If $\{\underline{v}(\mathcal{S}_k)\} \subset \mathcal{T}(\mathbf{R}^3)$ converges to $\underline{v}(\mathcal{S}, \theta) \in \overline{\mathcal{T}(\mathbf{R}^3)}$ in the measure theoretic sense, and $\mathcal{S}$ is C^0 embedded, then $\mathcal{S}$ is a Lipschitz surface.*

Proof. Assume initially that $\zeta \in \mathcal{S}$ is a good point for $\varepsilon \le \varepsilon_0$, where the notation is the same as above. In order to prove that $\mathcal{S}$ has a bilipschitz parameterization in a neighborhood of ζ it is enough to show that there exists $r \in (0, \sigma/64)$ so that for $i, j \in \{1, \cdots, N\}$ either

$$(*) \quad \Phi^j(D) \cap \Phi^i(D) \cap B_r(\zeta) = \varnothing \quad \text{or} \quad \Phi^i(D) \cap B_r(\zeta) = \Phi^j(D) \cap B_r(\zeta).$$

Assume this is not the case. Using the hypothesis that $\mathscr{S}$ is C^0 embedded in U, we can choose $r \in (0, \sigma/64)$ so that $\mathscr{S} \cap B_r(\zeta)$ is homeomorphic to a flat domain $R \subset \mathbf{R}^2$ via a map f, and $f(\zeta) \in \operatorname{int} \mathbf{R}$. We may also assume that r is small enough so that

$$\mathscr{S} \cap B_r(\zeta) = \bigcup_{\zeta \in \Phi^j(D)} \Phi^k(D) \cap B_r(\zeta),$$

and each one of the $\Phi^j(D) \cap B_r(\zeta)$ is connected. Since f is a homeomorphism and Φ^j is a bilipschitz map, $f(\Phi^j(D) \cap B_r(\zeta))$ is an open set in R for each j. Furthermore, $f(\zeta) \in \operatorname{int} R$ implies that there exists $\sigma > 0$ so that

$$B_\sigma(f(\zeta)) \cap \mathbf{R}^2 \subset \bigcap_{j=1}^{N} f(\Phi^j(D) \cap B_r(\zeta)).$$

The fact that $(*)$ does not hold for any $r > 0$ means that, for example, $\Phi^1(D_1) \cap B_r(\zeta) \neq \Phi^2(D_2) \cap B_r(\zeta)$, $\forall r > 0$. Since $\partial \Phi^j(D) \cap B_r(\zeta) = \varnothing$, there exists a sequence $\{\zeta_n^1\}_n \subset (\Phi^1(D) \cap B_r(\zeta)) \backslash (\Phi^2(D) \cap B_r(\zeta))$ or a sequence $\{\zeta_n^2\}_n \subset (\Phi^2(D) \cap B_r(\zeta)) \backslash (\Phi^1(D) \cap B_r(\zeta))$ converging to ζ. Thus for n large enough $f(\zeta_n^j) \in B_\sigma(f(\zeta)) \cap \mathbf{R}^2$ for $j = 1$ or $j = 2$, suppose $j = 1$; the $f(\zeta_n^1) \in f(\Phi^2(D) \cap B_r(\zeta))$ and hence $\zeta_n^1 \in \Phi^2(D) \cap B_r(\zeta)$ because f is a homeomorphism. This contradicts the choice of the $\{\zeta_n^1\}_n$. Hence, locally, $\mathscr{S}$ admits bilipschitz parameterizations away from finitely many bad points $\zeta_1, \cdots, \zeta_p$.

Assume now that for some $0 < \delta = \delta(\varepsilon_0) < \varepsilon_0$,

$$\lim_{\sigma \to 0} \liminf_{k \to \infty} \int_{\mathscr{S}_k \cap B_\sigma(0)} |A_k|^2 \, d\mathscr{H}^2 \geq \delta^2.$$

Nevertheless we claim that there exists a subsequence $\mathscr{S}_{k'}$ (denoted subsequently by $\mathscr{S}_k$) so that

$$\lim_{\sigma \to 0} \lim_{\theta \to 0} \liminf_{k \to \infty} \int_{\mathscr{S}_k \cap (B_\sigma(0) \backslash B_{\theta\sigma}(0))} |A_k|^2 \, d\mathscr{H}^2 \leq \varepsilon_0^2.$$

Otherwise we could find a decreasing sequence $\{\sigma_k\}_{k \geq 0}$ converging to 0 and so that for each $k \geq 0$ there exists n_k such that $\forall n \geq n_k$,

$$\int_{\mathscr{S}_n \cap (B_{\sigma_k}(0) \backslash B_{\sigma_{k+1}}(0))} |A_n|^2 \, d\mathscr{H}^2 \geq \frac{\delta^2}{2}.$$

Since $\{\mathscr{S}_k\} \subset \mathscr{T}(U)$, there exists $C > 0$ such that

$$\sup_{k\geq 0} \int_{\mathscr{S}_k \cap B_{\sigma_0}(0)} |A_k|^2 \, d\mathscr{H}^2 \leq C.$$

Let $k_0 > 0$ be large enough so that $\frac{1}{2}k_0\delta^2 \geq C$, and let $N_0 = \max_{0\leq i\leq k_0} n_i$. Then

$$\int_{\mathscr{S}_{N_0}\cap B_{\sigma_0}(0)} |A_{N_0}|^2 \, d\mathscr{H}^2 \geq \sum_{i=0}^{k_0} \int_{\mathscr{S}_{N_0}\cap(B_{\sigma_i}(0)\setminus B_{\sigma_{i+1}}(0))} |A_{N_0}|^2 \, d\mathscr{H}^2$$
$$\geq (k_0+1)\delta^2 > C,$$

which contradicts the assumption that $\sup \int_{\mathscr{S}_k\cap B_{\sigma_0}(0)} |A_k|^2 \leq C$. Therefore we can find a subsequence, denoted subsequently simply by $\{\mathscr{S}_n\}$ such that for fixed $\sigma > 0$ we have

$$\int_{\mathscr{S}_n\cap(B_\sigma(0)\setminus B_{\sigma/2^n}(0))} |A_n|^2 \, d\mathscr{H}^2 \leq \delta^2 < \varepsilon_0^2.$$

Let $n \geq 3$ and let $\mathscr{S}'$ denote any of the $\mathscr{S}_n$. Then the argument used in [12] to prove the Approximate Graphical Decomposition Lemma goes through, in the codimension-one case, if we replace $\mathscr{S}' \cap B_\sigma(0)$ by $\mathscr{S}' \cap (B_\sigma(0)\setminus B_{\sigma/8}(0))$ as long as

$$\int_{\mathscr{S}'\cap(B_\sigma(0)\setminus B_{\sigma/8}(0))} |A|^2 \, d\mathscr{H}^2 \leq \delta^2 < \varepsilon_0^2.$$

Namely there are pairwise disjoint sets $P_1, \cdots, P_N \subset \mathscr{S}'$ with

$$\sum_{j=1}^{N} \operatorname{diam} P_j \leq C\varepsilon^{1/2}\sigma,$$

$\sigma_1 \in (\frac{1}{8}\sigma, \frac{1}{4}\sigma)$ and $\sigma_2 \in (\frac{1}{2}\sigma, \sigma)$ such that for $i = 1, 2$, $\partial B_{\sigma_i}(0)$ intersects $\mathscr{S}'$ transversely, $\partial B_{\sigma_i}(0) \cap (\bigcup_j P_j) = \varnothing$, and

$$\mathscr{S}' \cap (B_{\sigma_2}(0)\setminus B_{\sigma_1}(0)) = \bigcup_{j=1}^{M} A_i(\sigma_1, \sigma_2),$$

where each $A_i(\sigma_1, \sigma_2)$ is topologically an annulus so that $\operatorname{diam} A_i(\sigma_1, \sigma_2) \geq C^{-1}\sigma_1$. Moreover there exist functions $u_i \in C^\infty(\overline{\Omega}_i, L_i^\perp)$ with L_i a plane in $\mathbf{R}^3$, and Ω_i a smooth bounded domain in L_i of the form $\Omega_i = \Omega_i^0\setminus(\bigcup_k d_{i,k})$, where Ω_i^0 is connected, and $d_{i,k}$ are pairwise disjoint closed discs in L_i which do not intersect $\partial\Omega_i^0$, with graph u_i connected, and with

$$\sup_{\Omega_i}\sigma^{-1}|u_i| + \sup_{\Omega_i}|Du_i| \le C\varepsilon^{1/6},$$
$$\operatorname{graph} u_i \cap (B_{\sigma_2}(0)\backslash B_{\sigma_1}(0)) \subset A_i(\sigma_1, \sigma_2),$$

where $A_i(\sigma_1, \sigma_2)\backslash \operatorname{graph} u_i$ is a union of a subcollection of the P_j, and each P_j is topologically a disc.

Now choose $\rho \in (\frac{1}{4}\sigma, \frac{1}{2}\sigma)$ so that $\partial B_\rho(0) \cap (\bigcup_{i,k} d_{i,k}) = \varnothing$, and

$$\int_{\partial B_\rho(0)\cap L_i} |D^2 u_i| \le \frac{C}{\sigma}\int_{\Omega_i\cap(B_{\sigma/2}(0)\backslash B_{\sigma/4}(0))} |D^2 u_i|^2 \le \frac{C}{\sigma}\int_{A_i(\sigma_1, \sigma_2)} |A|^2\, d\mathscr{H}^2,$$

where the second inequality comes from the fact that $|Du_i| \le \frac{1}{2}$.

Let $w_i \in C^\infty(L_i \cap \overline{B}_\rho(0))$ satisfy

$$\begin{cases} \Delta^2 w_i = 0 & \text{on } L_i \cap B_\rho(0), \\ w_i = u_i, \quad Dw_i = Du_i & \text{on } L_i \cap \partial B_\rho(0). \end{cases}$$

Then (see [11])

$$\int_{L_i\cap B_\rho(0)} |D^2 w_i|^2 \le C\rho\int_{\partial B_\rho(0)\cap L_i} |D^2 u_i|^2 \le C\int_{A_i(\sigma_1, \sigma_2)} |A|^2\, d\mathscr{H}^2.$$

In particular

$$\int_{\operatorname{graph} w_i} |\widetilde{A}_i|^2 \le C\int_{L_i\cap B_\rho(0)} |D^2 w_i|^2 \le C\int_{\Omega_i\cap(B_{\sigma/2}(0)\backslash B_{\sigma/4}(0))} |D^2 u_i|^2,$$

where $\widetilde{A}_i$ is the second fundamental form of graph w_i. Let

$$\widetilde{\mathscr{S}} = (\mathscr{S}'\backslash B_{\sigma_2}(0)) \cup \left(\bigcup_i^M (A_i(\sigma_1, \sigma_2)\backslash C_\rho^i)\right) \cup \left(\bigcup_i^M \operatorname{graph} w_i\right),$$

where C_ρ^i is the cylinder $(L_i \cap B_\rho(0)) \times (L_i)^\perp$. Then $\widetilde{\mathscr{S}}$ is a $C^{1,1}$ composite surface, satisfying

$$\int_{\widetilde{\mathscr{S}}\cap B_\sigma(0)} |\widetilde{A}|^2 \le \int_{\mathscr{S}'\cap(B_\sigma(0)\backslash B_{\sigma/8})(0)} |A|^2 + \sum_i \int_{\operatorname{graph} w_i} |\widetilde{A}_i|^2,$$

and

$$\int_{\widetilde{\mathscr{S}}\cap B_\sigma(0)} |\widetilde{A}|^2 \le C\delta^2 \le \frac{1}{2}\varepsilon_0^2,$$

for δ small enough.

We have constructed a new sequence $\{\widetilde{\mathscr{S}}_k\}$ of $C^{1,1}$ composite surfaces that converges to $\mathscr{S}$ in the measure theoretic sense and so that

$$\lim_{r\to 0}\left(\liminf_{k\to\infty}\int_{\widetilde{\mathscr{S}}_k\cap B_r(0)}|\widetilde{A}_k|^2\,d\mathscr{H}^2\right)\le \frac{1}{2}\varepsilon_0^2 .$$

Therefore we can find a sequence $\{\widehat{\mathscr{S}}_k\}\subset\mathscr{T}(U)$ converging to $\mathscr{S}$ in the measure theoretic sense and so that for all k

$$\int_{\widehat{\mathscr{S}}_k\cap B_\sigma(0)}|\widehat{A}_k|^2\le 2\int_{\widetilde{\mathscr{S}}_k\cap B_\sigma(0)}|\widetilde{A}_k|^2\le C\delta^2\le\varepsilon_0^2 .$$

Thus the origin is not a bad point for ε_0 with respect to this new sequence, and $\mathscr{S}$ admits a bilipschitz parameterization in a neighborhood of 0.

Remark. If $\zeta\in\mathscr{S}$ and $\Phi\colon D\subset\mathbf{R}^2\to\mathscr{S}\cap B_r(\zeta)$ is the bilipschitz parameterization constructed above, then from Theorem 3.1 it follows that Φ is a quasi-isometry in the sense that

$$\|(d\Phi)^*\circ(d\Phi)-\imath\|_{L^\infty(D)}\le C\varepsilon_0^2 .$$

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